Curve and its Jacobian

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We assume that K is algebraically closed.

Equivalent definition

Consider the **structure** on the set C defined by the $(4g+2)$-ary relation
$R(u_1, \ldots, u_g, u_{g+1}, v_1, \ldots, v_g, v_{g+1}, t_1, \ldots, t_g, s_1, \ldots, s_g)$
interpreted as
$u_1 + \ldots + u_g + u_{g+1} + s_1 + \ldots + s_g \equiv_{\text{linear}}$
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This relation is equivalent to the relation

$$x_1 + \ldots + x_g + x_{g+1} = y_1 + \ldots + y_g$$

for 0-degree divisors of the form $x_i = [u_i - v_i]$, $y_j = [t_j - s_j]$.

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Lemma. *C^J is beinterpretable with $(J;C,+)$ using parameters.*

Theorem by F.Bogomolov, M.Korotaev and Yu.Tschinkel:

Let $K = \mathbb{F}_p^{alg}$ *be the algebraic closure of a field of p elements,* $p > 3$ *and*

$$(J_1; C_1, +) \cong (J_2; C_2, +)$$

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The proof by BKT is very specific to the locally finite field $\mathbb{F}_p^{alg}$ and use the theory of profinite groups of automorphisms. The motivation comes from *anabelian geometry*.

Our Main Theorem.

Given algebraically closed fields K_1 *and* K_2, $C_1 = C_1(K_1)$, $C_2 = C_2(K_2)$, *assume*

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Then there is a field-isomorphism $\beta : K_1 \to K_2$ *inducing an isomorphism on pairs* $(J_1; C_1, +) \to (J_1'; C_1', +)$ *and a bijective isogeny* $\psi : J_1' \to J_2$ *such that*

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When the characteristic of the field is 0 the isogeny ψ *is an isomorphism of algebraic varieties.*

From model theoretic point of view

Lemma. C^J is a strongly minimal structure with a non-locally modular pregeometry.

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Restricted Trichotomy Conjecture (1983)
A strongly minimal structure M definable in an algebraically closed field K satisfies one of the following

- the pregeometry of M is trivial;
- the pregeometry of M is locally modular non-trivial;
- a field structure F is definable in M, moreover $F \cong_\beta K$ via a definable isomorphism β.

Theorem by E.Rabinovich (1988)

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Rabinovich' theorem implies the Main Theorem.

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2. The stabilizer of C in J is trivial.
3. C is in $\mathrm{dcl}(F)$ for F any ACF definable in C^J.

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2. Find a proof of Rabinovich' theorem based on the classification of Zariski geometries.