Factorization of Differential Operators and A Jordan-Hölder Theorem for Differential Algebraic Groups

(joint work with Phyllis Cassidy)

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Recent Developments in Model Theory

Oléron, June 5-11, 2011

Factorization

$(k,')$ a differential field, characteristic 0

$k[D]$ = ring of (ordinary) differential operators

$$= \{a_n D^n + \ldots + a_1 D + a_0 \mid a_i \in k\} \quad Da = aD + a'$$

<u>Ex.</u> $k = \mathbb{Q}(x)$

$$\begin{aligned} D^2 - 2D + 1 &= (D-1)\cdot(D-1) \\ &= (D - (1 - \frac{a}{ax+b}))\cdot(D - (1 + \frac{a}{ax+b})) \quad a, b \in \mathbb{Q} \end{aligned}$$

$V_1 = \mathrm{Soln}(y' - y = 0) = \{ce^x \mid c' = 0\}$

$V_2 = \mathrm{Soln}(y' - (1 + \frac{a}{ax+b})y = 0) = \{c(ax+b)e^x \mid c' = 0\}$

$\phi : V_1 \to V_2 \qquad \phi(v_1) = (ax+b)v_2$

Theorem. Let $L \in k[D] \backslash k$.

1) There exist irreducible $L_i \in k[D]$ such that

$$L = L_1 L_2 \cdots L_r.$$

2) If $L = M_1 M_2 \cdots M_s$, $M_i \in k[D]$ and irreducible, then

$$r = s$$

and, after a possible renumbering, $\exists R_i \in k[D]$ s.t.

$$R_i : \mathrm{Soln}_K(L_i(y) = 0) \to \mathrm{Soln}_K(M_i(y) = 0)$$

is a bijection and where K is any differentially closed field containing k.

For $k = \mathbb{Q}(x),\ x' = 1$, one can effectively compute such a factorization.

An example of Edmund Landau (1912)

$$\mathbb{C}(x, y), \Delta = \{\partial_x, \partial_y\}$$

$$\begin{aligned} L &= \partial_x^3 + x\partial_x^2\partial_y + 2\partial_x^2 + 2(x+1)\partial_x\partial_y + \partial_x + (x+2)\partial_y \\ &= (\partial_x + 1)(\partial_x + 1)(\partial_x + x\partial_y) \\ &= (\partial_x^2 + x\partial_x\partial_y + \partial_x + (x+2)\partial_y)(\partial_x + 1)\ . \end{aligned}$$

All factors are irreducible over any Δ-field containing $\mathbb{C}(x, y)$.

Is there a context in which we have a unique decomposition?

• Unique factorization of $n \in \mathbb{N} \Leftrightarrow$ Jordan-Hölder for groups applied to $\mathbb{Z}/n\mathbb{Z}$

• Solutions of $L(Y) = 0$ form a Differential Algebraic Group

What is a Jordan-Hölder Theorem for differential algebraic groups and what does it say about Landau's example?

Outline

- Differential Algebraic Groups
- Kolchin Polynomial and Differential Type
- Almost Simple and Isogenous Groups
- Jordan-Hölder Theorem
- Classification of Almost Simple Groups

Differential Algebraic Groups

(k, Δ) = differential field, char. 0, $\Delta = \{\partial_1, \ldots, .\partial_\ell\}$

$k\{y_1, \ldots, y_m\}$ = the ring of differential polynomials in m variables

Def. k is Differentially Closed if for any differential ideal
$I \triangleleft k\{y_1, \ldots, y_m\}$, $1 \notin I$, $\exists (z_1, \ldots, z_m) \in k^m$ s.t. $f(z_1, \ldots, z_m) = 0 \ \forall f \in I$.
= model of ℓ-DCF.

Def. $X \subset k^m$ is Δ-closed if there exist $f_1, \ldots f_r \in k\{y_1, \ldots, y_m\}$ such that

$$X = \{(z_1, \ldots, z_m) \in k^m \mid f(z_1, \ldots, z_m) = \ldots = f_r(z_1, \ldots, z_m) = 0\}$$

Can define Δ-affine varieties, Δ-morphism, Δ-abstract varieties

Δ-differential algebraic group = group object in category of Δ-abstract varieties

Def. A Δ-closed subgroup $G \subset \mathrm{GL}_n(k)$ is called a Linear Δ-Group.

Ex. 1 Any linear algebraic group $G(k) \subset \mathrm{GL}_n(k)$ is a linear Δ-group.

Ex. 2 If $G \subset \mathrm{GL}_n$ is an algebraic group, then $G(C) = G(k) \cap \mathrm{GL}_n(C)$ is a linear Δ-group. $C = \{c \in k \mid \partial(c) = 0 \ \forall \partial \in \Delta\}$.

Ex. 3 Subgroups of $(k, +) = \mathbb{G}_a(k) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$.

For $L_1, \ldots, L_m \in k[\partial_1, \ldots, .\partial_\ell]$

$\{u \in \mathbb{G}_a(k) \mid L_1(u) = \ldots = L_m(u) = 0\}$ is a linear Δ-group.

All Δ-subgroups of $\mathbb{G}_a(k)$ are of this form.

Def. G, H linear Δ-groups. A homomorphism $\phi : G \to H$ is a Δ-homomorphism if it is an everywhere defined rational differential map, i.e., the coordinate functions are everywhere defined quotients of differential polynomials.

Ex. $\Delta = \{\partial\}$

$$\phi : \mathbb{G}_m(k) \to \mathbb{G}_a(k)$$

$$\phi : u \mapsto \frac{\partial u}{u}$$

Thm. (Cassidy) A quasisimple linear Δ-group is Δ-isomorphic to $G(C')$ where

1. G is a quasisimple linear *algebraic* group defined over $\mathbb{Q}$, and
2. $C' = \{c \in k \mid \partial'(c) = 0 \ \forall \partial' \in \Delta'\}$ for some $\Delta' \subset k\Delta$, the k-span of Δ.

Quasisimple = normal Δ-subgroups are finite.

Kolchin Polynomial and Differential Type

P = prime differential ideal in $k\{y_1, \ldots, y_m\}$
$\Rightarrow k\{y_1, \ldots, y_m\}/P$ is a differential integral domain.

Let $k < \bar{y}_1, \ldots, \bar{y}_m >$ = quotient field of $k\{y_1, \ldots, y_m\}/P$

For $t \in \mathbb{N}$, let

$$k_t = k(\{\partial_1^{i_1} \cdots .\partial_\ell^{i_\ell} \bar{y}_j \mid i_1 + \ldots + i_n \leq t,\ 1 \leq j \leq m\}) \subset k < \bar{y}_1, \ldots, \bar{y}_m >$$

Prop. (Kolchin) There exists an integer valued polynomial $\omega_P(t)$ such that

$$\omega_P(t) = \text{tr. deg.}_k k_t \quad \text{for } t >> 0$$

We have $\deg \omega_P(t) \leq \ell$, $|\Delta| = \ell$ and

$$\omega_P(t) = a_\ell \binom{t+\ell}{\ell} + a_{\ell-1} \binom{t+\ell-1}{\ell-1} + \ldots + a_0, \quad a_i \in \mathbb{Z}$$

Def. The largest τ such that $a_\tau \neq 0$ is the differential type.
($\tau = -1$ if $\omega_P(t) = 0$).

Ex. $(k, \{\partial_1, \ldots, \partial_\ell\})$

$$P = (0) \subset k\{y_1, \ldots, y_m\}, \quad P \Leftrightarrow k^m$$

$$\omega_P(t) = m\binom{t+\ell}{\ell}$$

$$\Downarrow$$

$$\tau = \ell,\ a_\tau = m$$

Ex. $(k, \{\partial\})$

$$P = (L(y)) \subset k\{y\},\ L(y) \text{ linear}$$

$$\omega_P(t) = n = \text{the order of } L$$

$$\Downarrow$$

$$\tau = 0,\ a_\tau = \text{the order of } L$$

Meaning of τ and a_τ: *After a change of variables, the solutions of the equations in P are determined by a_τ arbitrary functions of τ variables*

Formalized as above by Kolchin - ideas already in Cartan, Einstein, ...

Almost Simple and Isogenous Goups

Given a a Δ-group G, we want a tower $G = G_1 \supset G_2 \supset \ldots \supset G_r = (0)$ where G_i/G_{i+1} are "simple".

<u>Ex.</u> $k \supset \mathbb{C}(x), \quad \Delta = \{\partial_x\}, \quad G = \mathbb{G}_a(k)$

$$H \lneq G \Rightarrow H = \{z \mid L(z) = 0\} \text{ for some } L \in k[\partial_x].$$
$$\text{For } H' = \{z \mid (\partial_x \circ L)(z) = 0\},\ H \lneq H' \lneq G.$$

Can never have a tower with *simple* quotients.

Note: G has type 1 and all proper subgroups have type 0.

<u>Def.</u> A Δ-group G is Almost Simple if for any *proper, normal* Δ-subgroup H, the differential type of H is *less than* the differential type of G.

Ex. $G = \mathbb{G}_a(k)$ is almost simple.

Ex. $|\Delta| = 1 \quad L \in k[\partial], L \neq 0, G = \{z \mid L(z) = 0\}$

$$G \text{ almost simple} \Leftrightarrow G \text{ simple} \Leftrightarrow L \text{ irreducible.}$$

Ex. For any (k, Δ), $L \in k[\partial_1, \ldots, .\partial_\ell]$

$$L = L_1 \circ L_2 \;\Rightarrow\; G = \{z \mid L(z) = 0\} \text{ is not almost simple.}$$
$$L \text{ irreducible} \not\Rightarrow\; G = \{z \mid L(z) = 0\} \text{ is almost simple.}$$

Prop. If G is a Δ-group then there exists a tower

$$G = G_1 \supset G_2 \supset \ldots \supset G_r = (0)$$

such that G_i/G_{i+1} is almost simple.

In what sense are the G_i/G_{i+1} unique?

Def. Let G_1, G_2 be Δ-groups. We say G_1 and G_2 are isogenous if there exists a Δ-group G and Δ homomorphisms

$$\begin{array}{ccccc} & & G & & \\ & \swarrow \phi_1 & & \phi_2 \searrow & \\ G_1 & & & & G_2 \end{array}$$

such that the $\tau(\ker \phi_i) < \tau(G)$.

The Jordan-Hölder Theorem

Def. A Δ-group G is strongly connected if for any proper normal Δ-subgroup $H \lhd G$, $\tau(G/H) = \tau(G)$.

Ex. Any Δ-subgroup of k defined by a single equation is strongly connected.

Theorem. If G is a strongly connected Δ-group, then there exists a tower of Δ-groups

$$G = G_1 \rhd G_2 \rhd \ldots \rhd G_r = (0)$$

such that G_i/G_{i+1} is almost simple. If

$$G = H_1 \rhd H_2 \rhd \ldots \rhd H_s = (0)$$

is another such tower, then $r = s$ and each G_i/G_{i+1} is isogenous to some H_j/H_{j+1}.

Ex. (Landau)

$$G: \quad \partial_x^3 + x\partial_x^2\partial_y + 2\partial_x^2 + 2(x+1)\partial_x\partial_y + \partial_x + (x+2)\partial_y$$
$$= (\partial_x + 1)(\partial_x + 1)(\partial_x + x\partial_y)$$
$$= (\partial_x^2 + x\partial_x\partial_y + \partial_x + (x+2)\partial_y)(\partial_x + 1)\ .$$

$$\begin{array}{lll} G & \qquad & G \\ \cup & & \cup \\ G_1 : (\partial_x + 1)(\partial_x + x\partial_y) & & H_1 : L_1, L_2 \\ \cup & & \cup \\ G_2 : \partial_x + x\partial_y & & H_2 : \partial_x + 1 \\ \cup & & \cup \\ (0) & & (0) \end{array}$$

$L_1 = x\partial_x{}^2\partial_y + x^2\partial_x\,\partial_y{}^2 - \partial_x{}^2 - \partial_x\,\partial_y + x^2\partial_y{}^2 - \partial_x - \partial_y - x\partial_y$
$L_2 = \partial_x{}^3 - x^2\partial_x\,\partial_y{}^2 + 3\,\partial_x{}^2 + 2\,x\partial_x\,\partial_y + 3\,\partial_x\,\partial_y - x^2\partial_y{}^2 + 2\,\partial_x + 2\,x\partial_y + 3\,\partial_y$
$\langle(\partial_x + 1)\rangle \cap \langle(\partial_x + x\partial_y)\rangle = \langle L_1, L_2\rangle$

$$G/G_1 \simeq H_2 \qquad G_1/G_2 \simeq G/H_1 \qquad G_2 \simeq H_1/H_2$$

Ex.

$$\begin{aligned} G: \quad & x\partial_x^3 - x^2\partial_x^2\partial_t - 2\partial_x^2 - x\partial_x\partial_t + x^2\partial_t^2 + 2\partial_t \\ & = (x\partial_x - x^2\partial_t - 2)(\partial_x^2 - \partial_t) \\ & = (x\partial_x^2 - x\partial_t - 2\partial_x)(\partial_x - x\partial_t) \end{aligned}$$

$$\begin{array}{ccc} G & \quad\quad & G \\ \cup & & \cup \\ G_1 : (\partial_x^2 - \partial_t) & & H_1 : (\partial_x - x\partial_t) \\ \cup & & \cup \\ (0) & & (0) \end{array}$$

G/G_1 isogenous to H_1 and G_1 isogenous to G/H_1 BUT $G_1 \not\simeq G/H_1$.

$$\begin{array}{ccc} G_1 : (\partial_x^2 - \partial_t) & \rightarrow & G/H_1 : (x\partial_x^2 - x\partial_t - 2\partial_x) \\ u & \stackrel{\phi}{\longmapsto} & \partial_x u - x\partial_t u \end{array}$$

Cartan: *Les groupes [G_1 et G/H_1] ne sont pas isomorphes holoédriques ...*

Classification of Almost Simple Δ-groups

We do not have a general classification!

Ex. $\Delta = \{\partial\}$

- $\mathbb{G}_m(k)$ and $\mathbb{G}_a(k)$ are almost simple.
- $\phi(u) = \frac{\partial_x u}{u}$ is an isogeny from $\mathbb{G}_m(k)$ onto $\mathbb{G}_a(k)$.

Ex.

- For each $r \in \mathbb{N}$, let

$$G_r \quad = \quad \{(u, v) \in \mathbb{G}_m(k) \times \mathbb{G}_a(k) \mid \frac{\partial_x(u)}{u} = \partial_x{}^r(y)\}$$

  $G_r(k)$ is almost simple.
- The map $\pi : G_r(k) \to \mathbb{G}_a(k)$ given by $\pi(u, v) = v$ is an isogeny.
- For $r \neq s$, $G_r(k)$ and $G_s(k)$ are not isomorphic.

Thm. Assume $|\Delta| = 1$ and let G be a nontrivial almost simple linear Δ-group.

1. G non-commutative $\Rightarrow$ there exists a quasisimple algebraic group H defined over $\mathbb{Q}$ such that

   a. G is Δ-isomorphic to $H(k)$, or
   b. G is Δ-isomorphic $H(C)$ where C are the Δ-constants.

2. G is commutative, then G is either

   a. Δ-isomorphic to $\mathbb{G}_a(C)$ or $\mathbb{G}_m(C)$, or
   b. isogenous to $\mathbb{G}_a(k)$. Furthermore, there is a classification (up to isomorphism) of groups isogenous to $\mathbb{G}_a(k)$.

- The proof of 1. uses Altinel/Cherlin (1998):

  G perfect, fin. Morley Rank + $G/Z(G)$ quasisimple, alg. $\Rightarrow$ G alg.

- In general, G a Δ-group $\not\Rightarrow$ $[G, G]$ closed, BUT for $|\Delta| \geq 1$

  G almost simple $\Rightarrow$ G is perfect (Freitag) and $G/Z(G)$ quasisimple, algebraic (Cassidy/Singer)
  ???? $\Rightarrow$????G algebraic

Ex. The Heat Equation. Let $\Delta = \{\partial_x, \partial_t\}$ and let

$$H = \{z \in \mathbb{G}_a(k) \mid \partial_t z = \partial_{xx} z\}$$

For each constant c, let

$$H_c = \{u \in \mathbb{G}_a(k) \mid \partial_t u = \partial_{xx} u + 2(c \tan cx)\partial_x u\}.$$

For all constant c, H_c is isogenous to H and if $c/d \notin \mathbb{Q}$, H_c and H_d are not isomorphic.

Indecomposability

• G an alg. gp., $\{Y_i\}_{i\in I}$ closed irreducible subsets with $e \in \cap_i Y_i$ then the group generated by the Y_i is

$$Y_{a(1)}^{\pm 1} \cdots Y_{a(n)}^{\pm 1} \text{ for some } a(1), \ldots, a(n).$$

and so is closed and connected.

• Generalized to groups of finite Morley rank (Zilber), superstable groups (Berline/Lascar), ...

Def. G a Δ-group. A definable subset X is α-indecomposable if for any Δ-closed subgroup H of G, either $\tau(X/H) \geq \alpha$ or $|X/H| = 1$.

Thm.(Freitag, 2011) If G is a Δ-group of type α, $\{Y_i\}_{i\in I}$ Δ-closed α-indecomposable subsets, $e \in \cap_i Y_i$, then the group generated by the Y_i Δ-closed

Cor. The commutator subgroup of a strongly connected Δ-group is Δ-closed. Almost simple Δ-groups are perfect.

Final Comments

- Jordan-Hölder Theorem true for all differential algebraic groups.
- Questions:
  - What are are the almost simple groups in general?
  - Understand isogeny classes, e.g., what is the isogeny class of the Heat Equation?
- Computation