Distal NIP theories

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NIP Theories

Definition

A formula $\phi(x; y)$ has the independence property if one can find some infinite set B such that for every $C \subset B$, there is y_C such that for $x \in B$,

$$\phi(x; y_C) \iff x \in C.$$

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Example

- *Stable theories,*
- *o-minimal,*
- $\mathbb{Q}_p$,
- *ACVF*

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It seems reasonable to look for 'stable parts' and 'order-controlled parts' of *NIP* structures or of types in them.

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- There are no (non-constant) totally indiscernible sequences.

Problem : This condition is not stable under going from M to M^{eq}.

Definition

The indiscernible sequence $I = I_1 + I_2 + I_3$ is *distal* if whenever

$$\left.\begin{array}{lllll} I_1 + & a & + I_2 & & + I_3 \\ I_1 & & + I_2 + & b & + I_3 \end{array}\right\} \text{ are indiscernible,}$$

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Definition

The theory T is distal if all indiscernible sequences are distal.

Remark

T is distal if and only if T^{eq} is distal.

Theorem

(T is NIP) The following are equivalent:

- *T is distal,*
- *For any two invariant types p_x and q_y, if $p_x \otimes q_y = q_y \otimes p_x$, then p_x and q_y are orthogonal,*
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Theorem

It is enough to check any one of these conditions in dimension 1.

Example

O-minimal theories and the p-adics are distal.

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For stable theories, it gives the usual non-forking relation. For distal theories, it is a trivial notion.

Theorem

Assume that $I_1 + I_2 + I_3$ *is an indiscernible sequence and* $I_1 + I_3$ *is indiscernible over* A*. Let* $\phi(x) \in L(A)$*, then*

$$\{b \in I_2 : \models \phi(b)\}$$

is finite or co-finite in I_2*.*

Thank you.