Finding something real in Zilber's Field

Ahuva Shkop

Center for Advanced Studies - Ben Gurion University

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Definition

- A *E-ring* is a $\mathbb{Q}$-algebra R with no zero divisors, together with a homomorphism $\exp : \langle R, +\rangle \to \langle R^*, \cdot\rangle$.
- A *partial E-ring* is a $\mathbb{Q}$-algebra R with no zero divisors, together with a $\mathbb{Q}$-linear subspace $A(R)$ of R and a homomorphism $\exp_R : \langle A(R), +\rangle \to \langle R^*, \cdot\rangle$. $A(R)$ is then the domain of $\exp_R$.
- An *E-field* is an E-ring which is a field.
- We say S *is a partial E-ring extension of* R if R and S are partial E-rings, $R \subseteq S$, and for all $r \in A(R)$, $\exp_S(r) = \exp_R(r)$.

Consider the language $\mathcal{L}_{\exp} = \{+, \cdot, 0, 1, \exp\}$, where $+, \cdot$ are binary functions, exp is a unary function, and $0, 1$ are constants. E-rings and E-fields are then $\mathcal{L}_{\exp}$-structures.

Examples

1. $\langle \mathbb{Q}, \{0\}, \exp \rangle$ where $\exp(0) = 1$ is a partial E-field.
2. $\langle \mathbb{Q}, +, \cdot, 0, 1, \exp \rangle$ where $\exp(q) = 1$ for all $q \in \mathbb{Q}$ is an E-field.
3. $\mathbb{R}_{exp} = \langle \mathbb{R}, +, \cdot, 0, 1, e^x \rangle$ is a real closed E-field.
4. $\mathbb{C}_{exp} = \langle \mathbb{C}, +, \cdot, 0, 1, e^z \rangle$ is an algebraically closed E-field.
5. If R is an E-ring and X a set of indeterminates, then $\langle R[X], R, \exp_R \rangle$ is a partial E-ring.
6. Zilber's pseudoexponential field is an exponential field.

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Motivation

Open questions about $\mathbb{C}_{\text{exp}}$:

- Is $\mathbb{C}_{\text{exp}}$ quasiminimal, i.e. is every definable set countable or co-countable?
- Are there any automorphisms of $\mathbb{C}_{\text{exp}}$ aside from the identity and complex conjugation?
- Does $\mathbb{C}_{\text{exp}}$ satisfy Schanuel's conjecture?

Conjecture (Schanuel)

If $z_1, \ldots, z_n \in \mathbb{C}$, then the transcendence degree over $\mathbb{Q}$ of $(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$ is at least the $\mathbb{Q}$ linear dimension of $(z_1, \ldots, z_n)$.

Zilber's Approach

In 2004, Zilber constructed the class of pseudoexponential fields. A pseudoexponential field, K, satisfies the following list of axioms:

1. $K \models ACF_0$
2. $\exp : (K, +) \to (K, \cdot)$ is a surjective homomorphism.
3. There is some transcendental ν so that $\ker(\exp_K) = \nu\mathbb{Z}$.
4. Schanuel property: If $a_1, \ldots, a_n \in K$ are $\mathbb{Q}$-linearly independent, then $td_{\mathbb{Q}}(\bar{a}, \exp(\bar{a})) \geq n$.
5. Exponential Closure: If $V \subseteq G_\alpha(K)$ is irreducible, rotund, and free, then for any finite $A \subset K$ there is $(a_1, \ldots, a_\alpha, \exp(a_1), \ldots, \exp(a_\alpha)) \in V$, a generic point in V over A.
6. Countable Closure: If $V \subseteq G_\alpha(K)$ is irreducible, rotund, free, and of dimension n, then for any finite $A \subset K$, the set of points $(\bar{a}, \exp(\bar{a})) \in V$ generic over A is countable. Equivalently, the Schanuel closure of a finite set is countable.

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Zilber proved the following two theorems:

Theorem

For κ uncountable, there is a unique pseudoexponential field of size κ and it has 2^κ-many automorphisms. Furthermore, pseudoexponential fields are quasiminimal.

Theorem

$\mathbb{C}_{exp}$ satisfies the countable closure axiom.

This leads to the following intriguing question:

Is $\mathbb{C}_{\text{exp}}$ the unique pseudoexponential field of size continuum?

Note: If Zilber's conjecture is true, then there are real closed subfields of pseudoexponential fields. Indeed, there must be an isomorphic copy of $\mathbb{R}_{\text{exp}}$ contained in the pseudoexponential field of size continuum.

Theorem

For $\mathcal{K}$ a pseudoexponential field, there are continuum many non-isomorphic (as fields) countable real closed exponential subfields of $\mathcal{K}$.

This is also true in $\mathbb{C}_{\text{exp}}$.

Free extensions and $K[X]^E$

Definition

Let R be a partial E-ring. We say $R' \supseteq R$ is a *free partial E-ring extension* of R if

- R' is a partial E-ring extension of R.
- The domain of $\exp_{R'}$ contains R.
- If $\{a_1, \ldots, a_n\} \subset R$ is $\mathbb{Q}$-linearly independent over $A(R)$, then $\{\exp(a_1), \ldots, \exp(a_n)\} \subset R'$ is algebraically independent over R.
- There is no partial E-subring of R' satisfying these conditions.

Note that the last condition guarantees that $A(R') = R$.

Main Lemmas

Lemma

Suppose R is a formally real partial E-ring. Then, R' is formally real.

Proof.

Look at the board! □

Consider a chain of subrings of $\mathcal{K}$ of the following form.

$$Q_0 \hookrightarrow Q_1 \hookrightarrow Q_2 \hookrightarrow \cdots$$

where $Q_0 \subseteq \mathbb{Q}^{alg}$ and $[Q_i \cup \exp(Q_i)] \subseteq Q_{i+1} \subseteq [Q_i \cup \exp(Q_i)]^{alg}$.

Lemma

Then $[Q_i \cup \exp(Q_i)] \cong Q_i'$.

Main Lemmas

Corollary

If Q_i is formally real, then $[Q_i \cup \exp(Q_i)]$ is formally real.

Consider the chain

$$Q_0 \hookrightarrow Q_1 \hookrightarrow Q_2 \hookrightarrow \cdots$$

where $Q_0 = \mathbb{Q}^{rc}$ and Q_{i+1} is a real closure of $[Q_i \cup \exp(Q_i)]$. Then the union is a real closed exponential subfield of $\mathcal{K}$.

Note: We can do this so that the exponential maps is order preserving.

What we really used: We used the fact that the ambient exponential field is algebraically closed and satisfies Schanuel's conjecture.

Adding Logs

Theorem

Let K be an algebraically closed exponential field with a surjective exponential map which satisfies Schanuel's conjecture. Then there are continuum many non-isomorphic (as fields) countable real closed exponential subfields of K whose exponential map is surjective onto the positive elements.

Note: This can be done so that the exponential map is still order preserving.

Proof Idea

$$Q_0 \hookrightarrow Q_1 \hookrightarrow \widetilde{Q}_1 \hookrightarrow Q_2 \hookrightarrow \widetilde{Q}_2 \hookrightarrow Q_3 \hookrightarrow \widetilde{Q}_3 \cdots$$

$$Q_0 = \mathbb{Q}^{rc}$$

$$Q_{i+1} = \widetilde{Q}_i[\exp(\widetilde{Q}_i)]^{rc} = \widetilde{Q}_i[E_{i+1}]^{rc}$$

$$\widetilde{Q}_{i+1} = Q_{i+1}[\log(Q_{i+1})]^{rc} = Q_{i+1}[L_{i+1}]^{rc}$$

$$\mathrm{dom}(\exp_{\widetilde{Q}_i}) = \mathbb{Q}+\mathrm{span}(\widetilde{Q}_{i-1} \cup L_i)$$
$$\mathrm{img}(\exp_{Q_i}) = \mathbb{Q}\cdot\mathrm{span}(Q_{i-1} \cup E_i)$$

2 Main Lemmas

Lemma

Suppose $\bar{a} \subseteq_{fin} \widetilde{Q}_i$ *is* $\mathbb{Q}$*-linearly independent over* $\widetilde{Q}_{i-1} \cup L_i$*. Then* $\exp(\bar{a})$ *is algebraically independent over* $\widetilde{Q}_i$*.*

Lemma

Suppose $\bar{a} \subseteq_{fin} Q_i^{>0}$ *is* $\mathbb{Q}$*- multiplicatively independent over* $Q_{i-1} \cup E_i$*. Then* $\log(\bar{a})$ *is algebraically independent over* Q_i*.*

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