

On the generalized Mordell-Lang conjecture over function fields of positive characteristic

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The generalized Mordell-Lang conjecture in char. 0

The following theorem is the result of the work of Faltings, Serre, Hindry and McQuillan:

Theorem (generalized Mordell-Lang conjecture)

Let A be an abelian variety over \mathbb{C} . Let $X \hookrightarrow A$ be a closed subvariety, which is not the translate of an abelian subvariety of A . Let $\Gamma \subseteq A(\mathbb{C})$ be a group such that $\Gamma \otimes \mathbb{Q}$ is a finite-dimensional \mathbb{Q} -vector space. Then $X \cap \Gamma$ is not Zariski dense in X .

The special case of the generalized Mordell-Lang conjecture where Γ is a group of torsion points (resp. is a finitely generated group) is referred to as the Manin-Mumford conjecture (resp. the Mordell-Lang conjecture).

Special cases

The following are consequences of GML.

Let C be a curve of genus ≥ 2 defined over a number field K .

Theorem (Mordell conjecture; Faltings [1984])

$C(K)$ is finite.

Theorem (Manin-Mumford conjecture for C ; Raynaud [1983])

Let $P \in C(\overline{K})$. Then the set

$$\{Q \in C(\overline{K}) \mid \exists n \in \mathbb{N}^* \text{ such that } nQ - nP \text{ is principal}\}$$

is finite.

Positive characteristic

It is natural to wonder if the analog of GML is true in positive characteristic. The following is known.

Let K be an algebraically closed field of char. $p > 0$. Let A be an abelian variety over K . Suppose that A has no factors, which are defined over a finite field.

Let $X \hookrightarrow A$ be a closed subvariety of A . Suppose that X is not the translate of an abelian subvariety of A .

Finally, let $\Gamma \subseteq A(K)$ be a group st $\Gamma \otimes \mathbb{Z}_p$ is a \mathbb{Z}_p -module of finite rank.

Theorem (Hrushovski [1996])

$X \cap \Gamma$ is not Zariski dense in X .

Around GML in positive characteristic

E. Hrushovski's proof of GML in positive characteristic relies heavily on model-theoretic methods. More precisely, his proof relies on a trichotomy theorem for the theory of separably closed fields with a finite p -basis. To establish this trichotomy, he needs his work with B. Zilber on Zariski geometries.

Here is what was already proven some years ago by algebraic methods:

- the case where A is ordinary (Abramovich-Voloch [1996], Pillay-Ziegler [2003]);
- the case where X is a curve and K is of transcendence degree 1 over \mathbb{F}_p (Grauert-Samuel [1966], Abramovich-Voloch [1996], Buium-Voloch [1996])

From Manin-Mumford to generalized Mordell-Lang

The aim of this talk is to sketch an algebraic proof of the following statement :

In positive characteristic, the Manin-Mumford conjecture implies the generalized Mordell-Lang conjecture. (*)

Combining this proof with the existing algebraic proof of the Manin-Mumford conjecture (see Pink-R. [2004]), one obtains an algebraic proof of the Mordell-Lang conjecture in positive characteristic.

The idea that the reduction (*) may be tractable was suggested by A. Pillay in a talk he gave in Paris in December 2010. He also outlined a model-theoretic approach to (*).

Jet schemes

Let U be a smooth curve over $\bar{\mathbb{F}}_p$.

Let \mathcal{X} be a scheme of over U .

We write $J^n(\mathcal{X}/U)$ for the n -th jet scheme of \mathcal{X} over U .

$J^n(\mathcal{X}/U)$ is a U -scheme, which depends covariantly on \mathcal{X} .

For our purpose, the most interesting property of jet schemes is the following :

- For any closed point $u \in U$, there is a natural identification

$$J^n(\mathcal{X}/U)(\kappa(u)) \simeq \mathcal{X}(u_n),$$

where $u_n \hookrightarrow U$ is the n -th infinitesimal neighborhood of u in U .

Properties of jet schemes

Here are further properties of jet schemes that we shall need :

- There are affine smooth "forgetful" morphisms

$$\cdots \rightarrow J^n(\mathcal{X}/U) \rightarrow J^{n-1}(\mathcal{X}/U) \rightarrow \cdots \rightarrow J^1(\mathcal{X}/U) = \mathcal{X}.$$

- There is a splitting map

$$\sigma_n : \mathcal{X}(U) \rightarrow J^n(\mathcal{X}/U),$$

which is natural in \mathcal{X} but does not arise from a scheme morphism.

Properties of jet schemes II

- If $\mathcal{X} = \mathcal{A}$ is an abelian scheme over U then the forgetful morphism $J^n(\mathcal{A}/U) \rightarrow J^{n-1}(\mathcal{A}/U)$ is a morphism of group schemes and its kernel is a vector bundle.
- There is a U -morphism

$$"p^{n-1}" : \mathcal{A} \rightarrow J^n(\mathcal{A}/U)$$

such that the composition of " p^{n-1} " with the forgetful morphism $J^n(\mathcal{A}/U) \rightarrow \mathcal{A}$ is the morphism p^{n-1} .

- $p^{n-1} \circ \sigma_n = \sigma_n \circ p^{n-1} = "p^{n-1}"$ on $\mathcal{A}(U)$.

The critical and exceptional schemes of a subscheme of an abelian scheme

Let now \mathcal{A}/U be an abelian scheme and $\mathcal{X} \hookrightarrow \mathcal{A}$ be a closed integral subscheme. We let

$$\text{Crit}^n(\mathcal{X}) := J^n(\mathcal{X}/U) \cap p^{n-1}(\mathcal{A})$$

The schemes $\text{Crit}^n(\mathcal{X})$ are proper over U and there are natural finite morphisms

$$\cdots \rightarrow \text{Crit}^n(\mathcal{X}) \rightarrow \text{Crit}^{n-1}(\mathcal{X}) \rightarrow \cdots \rightarrow \text{Crit}^1(\mathcal{X}) = \mathcal{X}$$

Let

$$\text{Exc}(\mathcal{X}) := \bigcap_{n \geq 1} \text{Im}(\text{Crit}^n(\mathcal{X}) \rightarrow \mathcal{X})$$

The non-density of the exceptional scheme

Let L be the function field of U .

Suppose that $\mathcal{A}_{\bar{L}}$ has no factors, which are defined over a finite field. Suppose that $\mathcal{X}_{\bar{L}}$ is not a union of translates of abelian subvarieties of $\mathcal{A}_{\bar{L}}$.

Proposition (non-density of Exc)

If the Manin-Mumford conjecture holds for $\mathcal{X}_{\bar{L}}$, then the set $\text{Exc}(\mathcal{X})$ is not Zariski dense in \mathcal{X} .

We shall sketch the proof later.

The exceptional properties of $\mathcal{X}(U)$

We contend that for $n \gg 1$, we have an inclusion

$$\mathcal{X} \cap p^n \mathcal{A}(U) \subseteq \text{Exc}(\mathcal{X})$$

Indeed, using the properties of jet schemes listed above, we get

$$\mathcal{X} \cap p^{n-1} \mathcal{A}(U) \subseteq \text{Im}(\text{Crit}^n(\mathcal{X}) := J^n(\mathcal{X}/U) \cap p^{n-1}(\mathcal{A}) \xrightarrow{\text{forgetful}} \mathcal{X})$$

for any $n \geq 1$.

Application to the Mordell-Lang conjecture

Granting the proposition on the non-density of Exc , we deduce that there is $n_0 = n_0(\mathcal{A}, \mathcal{X})$ such that the set

$$\mathcal{X} \cap p^{n_0} \mathcal{A}(U)$$

is not dense in \mathcal{X} .

Now suppose that $\mathcal{X}(U)$ is dense. Then for some $Q \in \mathcal{A}(U)$, the set

$$(\mathcal{X} + Q)(U) \cap p^{n_0} \mathcal{A}(U)$$

is dense in $\mathcal{X} + Q$, because $\mathcal{A}(U)$ is a finitely generated group (Mordell-Weil theorem).

Thus, the Mordell-Lang conjecture for \mathcal{X} would be proven if we knew that $n_0(\mathcal{A}, \mathcal{X} + Q) = n_0(\mathcal{A}, \mathcal{X})$.

Application to the Mordell-Lang conjecture II

In order to circumvent this problem, we enlarge the base-scheme U . Choose a closed point $u \in U$ and consider the spectrum of the completion

$$\widehat{U} := \text{Spec } \widehat{\mathcal{O}}_u$$

of the local ring of U at u .

If $\mathcal{X}(U)$ is assumed dense, there is a $Q \in \mathcal{A}(\widehat{U})$ such that the set

$$(\mathcal{X} + Q)(\widehat{U}) \cap p^{n_0} \mathcal{A}(\widehat{U})$$

is dense in $\mathcal{X} + Q$ for any $n_0 \geq 0$.

We would like to apply the reasoning of the last page to $\mathcal{X} + Q$, so we let $\mathcal{X} := \mathcal{X} + Q$ and we view \mathcal{X} and \mathcal{A} as \widehat{U} -schemes.

Application to the Mordell-Lang conjecture III

One problem now is that the jet schemes $J^n(\bullet)$, as well as Crit^\bullet and $\text{Exc}(\bullet)$ only naturally exist as formal schemes over \widehat{U} .

Nevertheless, by a theorem of Grothendieck ("Grothendieck's GAGA"), the schemes Crit^\bullet and $\text{Exc}(\bullet)$ have a unique underlying \widehat{U} -scheme because they are projective over \widehat{U} .

Furthermore, the proposition on the non-density of Exc still holds over \widehat{U} . This contradiction finishes the proof of the Mordell-Lang conjecture.

The non-density of $\text{Exc}(\mathcal{X})$

We now turn to the scheme $\text{Exc}(\mathcal{X})$. In order to show that $\text{Exc}(\mathcal{X})$ is not dense in \mathcal{X} , it is sufficient to show that $\text{Exc}(\mathcal{X})_u$ is not Zariski dense in \mathcal{X}_u .

Looking at the definitions, we see that this last statement is equivalent to the

Proposition ("generic non-liftability")

There is a natural number n_0 such that the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to } \mathcal{X}(u_{n_0}) \cap p^{n_0-1}\mathcal{A}(u_{n_0})\}$$

is not Zariski dense in \mathcal{X}_u .

On non-density of $\text{Exc}(\mathcal{X})$, II

The proposition on generic non-liftability was already proven by Buium-Voloch when \mathcal{X} is a relative curve of genus ≥ 2 and $\mathcal{A} = \text{Jac}(\mathcal{X}/\widehat{U})$. They showed that $n_0 = 2$ works in this situation.

The proof of the general form of the proposition is based on a galois-theoretic argument.

Let $\text{Frob} : \widehat{U} \rightarrow \widehat{U}$ be some power of the Frobenius automorphism of \widehat{U} . If this power is sufficiently big, we may assume that Frob acts on \mathcal{X} and \mathcal{A} .

Let g be the relative dimension of \mathcal{A} over \widehat{U} . On $\mathcal{A}(\kappa(u))$, the automorphism Frob satisfies a Galois equation $R(\text{Frob}) = 0$, where

$$R(T) = T^{2g} - (a_{2g-1} T^{2g-1} + \dots + a_1 T + a_0)$$

is a polynomial with integer coefficients.

On non-density of $\text{Exc}(\mathcal{X})$, III

Proposition

There is a closed subscheme $\mathcal{Y} \hookrightarrow \mathcal{X}$, such that:

- 1. the torsion points are dense in the generic fibre of \mathcal{Y} ;*
- 2. if $W \in \mathcal{X}(u_n)$ satisfies $R(W) = 0$ then $W \in \mathcal{Y}(u_n)$.*

Note that $\mathcal{X}_u \hookrightarrow \mathcal{Y}$. The scheme \mathcal{Y} will be highly nilpotent in general.

Corollary

If Manin-Mumford holds for \mathcal{X} , then there is an $n_0 \in \mathbb{N}$ st the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to a } W \in \mathcal{X}(u_{n_0}) \text{ satisfying } R(W)=0\}$$

is not Zariski dense.

On non-density of $\text{Exc}(\mathcal{X})$, IV

To conclude the proof of the non-density of $\text{Exc}(\mathcal{X})$, notice that the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to a } W \in \mathcal{X}(u_{n_0}) \text{ satisfying } R(W) = 0\}$$

contains the set

$$\{P \in \mathcal{X}(\kappa(u)) \mid P \text{ lifts to } \mathcal{X}(u_{n_0}) \cap p^{n_0-1}\mathcal{A}(u_{n_0})\}$$

because the morphism " p^{n_0-1} " commutes with Frob.