Connected components, universal covers and the Lascar group

Anand Pillay

University of Leeds

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Introduction I

- I give examples of definable groups G (in a saturated model) such that $G^{00} \neq G^{000}$. (Joint with A. Conversano [CPI])
- The basic example is (a saturated elementary extension of) the universal cover of $SL_2(\mathbb{R})$, in the group language, but there are closely related semialgebraic examples.
- By a standard construction we obtain "new" examples of non G-compact first order theories, i.e. where Lascar strong types do not coincide with compact (or KP) strong types.
- But maybe the latter are not so new after all: the examples in [CLPZ] were (in hindsight) based on covers of the circle (in a suitable language) and the fundamental group of $SL_2(\mathbb{R})$ coincides with that of a maximal compact $SO_2(\mathbb{R})$.

Introduction II

- Fix a $\emptyset$-definable group G in a saturated model of a theory T, and a small set A of parameters.
- G_A^0 is the intersection of all A-definable subgroups of G of finite index.
- G_A^{00} is the smallest type-definable over A subgroup of G of bounded index (i.e. index at most $2^{|A|+|T|}$).
- G_A^{000} is the smallest subgroup of G of bounded index which is $Aut(\bar{M}/A)$-invariant (equivalently whether or not $g \in G_A^{000}$ depends only on $tp(g/A)$).
- These are all normal subgroups of G and we have $G \geq G_A^0 \geq G_A^{00} \geq G_A^{000}$.
- In the cases we study (i.e. T has NIP), they are independent of the choice of A, and we just write G^0, G^{00}, G^{000}.

Introduction III

- Although the model-theoretic definitions of these "connected components" are similar (and in for example stable theories they coincide) the "typical" examples have very different mathematical flavours.
- G^0 is "like" the connected component (of identity) in the topological sense and agrees with it for groups G definable in $\mathbb{C} \models ACF$, or $\mathbb{R} \models RCF$.
- G^{00} is "like" the subgroup of "infinitesimals" and $G \to G/G^{00}$ "like" the standard part map.
- But always G/G^{00} has the structure of a compact Hausdorff topological group (via the logic topology), so its mathematical status is clear.
- The mathematical meanings of G^{000} and the quotient G^{00}/G^{000} are unclear. The latter could/should be viewed as an object of descriptive set theory or even noncommutative geometry. Discussed later.

Main example I

Theorem 0.1

Let $(G\cdot)$ be a saturated elementary extension of $(\widetilde{SL_2(\mathbb{R})}, \cdot)$. Then $G = G^{00}$ and G/G^{000} is (naturally) isomorphic to $\widehat{\mathbb{Z}}/\mathbb{Z}$.

I will briefly describe (components of) the proof.
First note that we have a (definable) exact sequence

$$1 \to \Gamma \to G \to SL_2(K) \to 1$$

for K a saturated RCF and Γ saturated elementary extension of $\mathbb{Z}$ (at least as a group).

Main example II

- (1) $[G, G]$ maps onto $SL_2(K)$ (as latter is perfect).
- (2) $[G, G] \cap \Gamma = \mathbb{Z}$ (to be discussed later).
- (3) $[G, G]$ is perfect (by (1) and (2) and perfectness of $\widetilde{SL_2(\mathbb{R})}$).
- (4) $[G, G] \subseteq G^{000}$. (By (2), (3) and abstract simplicity of $SL_2(K)$).
- (5) $G^{000} \cap \Gamma \geq \Gamma^0 = \cap_n n\Gamma$.
  This uses fact (to be seen later) that the only structure induced on Γ is its group structure, which we write additively.
- (6) $\Gamma^0 \cdot [G, G] = G^{000}$
  By (4) and (5) (to get $\subseteq$) together with $\Gamma^0 \cdot [G, G]$ being "invariant" and having bounded index in G.

Main example III

- (7) $G = G^{00}$.
  Proof. Note that $G^{000} \cap \Gamma = \Gamma^0 \cdot \mathbb{Z}$, using (2) and (6), so $G^{00} \cap \Gamma$ contains $\Gamma^0 \cdot \mathbb{Z}$. By denseness of $\mathbb{Z}$ in $\widehat{\mathbb{Z}} = \Gamma/\Gamma^0$ and type-definability of Γ^0, $G^{00} \cap \Gamma = \Gamma$, so by (6) and (1) we obtain (7) above.
- (8) $G/G^{000} = \hat{\mathbb{Z}}/\mathbb{Z}$.
  Proof. As both G and G^{000} project onto $SL_n(K)$, the exact sequence + (6) yields that $G/G^{000} = \Gamma/(\Gamma^0 \cdot \Gamma \cap [G, G])$ which equals $\Gamma/\Gamma^0 \cdot \mathbb{Z}$ which equals $\widehat{\mathbb{Z}}/\mathbb{Z}$ as required.

Main example IV

- The points (2) and (5) above depend on a canonical interpretation (with parameters) of $\widetilde{SL_2(\mathbb{R})}$ in the two sorted structure $((\mathbb{Z},+),(SL_2(\mathbb{R}),\cdot))$, using a certain "definable" cocycle $h : SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \to \mathbb{Z}$ (here with values $0,1$), described in a general context in [HPP].
- Then $\widetilde{SL_2(\mathbb{R})}$ is canonically isomorphic to $\mathbb{Z} \times SL_2(\mathbb{R})$ equipped with the group operation $*$ where $(a,x)*(b,y) = (a+b+h(x,y),xy)$.
- h could also be deduced from the obvious cocycle corresponding to the interpretation of the universal cover of $SO_2(\mathbb{R})$ in $((\mathbb{Z},+),(\mathbb{R},+,\times))$.

The semialgebraic and o-minimal contexts. I

- I first give a semialgebraic example with $G^{00} \neq G^{000}$.
- Fix an infinite cyclic subgroup $\langle \alpha \rangle$ of $SO_2(\mathbb{R})$ (necessarily dense).
- Use the cocycle h from the previous page to define a group operation $*$ on $SO_2(\mathbb{R}) \times SL_2(\mathbb{R})$ by: $(a, x) * (b, y) = (a + b + h(x, y)\alpha, xy)$. (Definable in $(\mathbb{R}, +, \cdot)$.)
- Let $(G, .)$ be a saturated elementary extension. Then a similar analysis to that in the proof of Theorem 0.1 yields:

Theorem 0.2
$G = G^{00}$ and G/G^{000} is (naturally) isomorphic to $SO_2(\mathbb{R})/\langle \alpha \rangle$.

The semialgebraic and o-minimal contexts. II

- I point out now that more or less the only way that G^{00} can be different from G^{000} for G definable in a (saturated) o-minimal expansion of a real closed field, is as in the above example.
- Fix a saturated o-minimal expansion $\bar{M}$ of RCF (or just a saturated real closed field) and G definable in $\bar{M}$. Assume G definably connected ($G = G^0$).
- There is then a unique maximal definable quotient D (maybe trivial) of G with the properties that there is a definable exact sequence $1 \to \Gamma \to D \to D_1 \to 1$ such that
  (i) Γ is definably connected, definably compact and central in D, and
  (ii) D_1 is definably connected, semisimple and strictly non definably compact (which amounts to saying that D_1 is semialgebraic, definable over $\mathbb{R}$, and $D_1(\mathbb{R})$ is an almost direct product of simple non compact Lie groups).

The semialgebraic and o-minimal contexts. III

With the above notation we have the following, appearing in [CPII], but no proof is given here.

Theorem 0.3
(i) G^{00}/G^{000} is (naturally) of the form A/Λ for A some, possibly trivial, connected commutative compact Lie group and Λ a finitely generated dense subgroup of A.
(ii) Moreover $G^{00}/G^{000} = D^{00}/D^{000}$.
(iii) Moreover A is (naturally) a (closed connected) subgroup of Γ/Γ^{00}, and Λ a quotient of the fundamental group of the semisimple Lie group $D_1(\mathbb{R})$.
(iv) Moreover any quotient of a connected commutative compact Lie group by a finitely generated dense subgroup can occur as G^{00}/G^{000} for some G.

Borel equivalence relations. I

- The material here is joint with K. Krupinski.
- In the above I mentioned the "naturality" of an isomorphism between G^{00}/G^{000} and $\widehat{\mathbb{Z}}/\mathbb{Z}$ (or A/Λ).
- But it is unclear what this means (as opposed to saying that G/G^{00} is isomorphic to S_1 say, where we mean as topological groups).
- One option, mentioned also in [CLPZ] but not explored much, is to plug into the theory of Borel equivalence relations from descriptive set theory.
- Assuming that everything around (theory, parameter set,..) is countable, then G/G^{000} (as well as the subgroup G^{00}/G^{000}) can be viewed as the quotient of a (subspace of a) type space over a countable model M_0, by a Borel, in fact K_σ, equivalence relation E.

Borel equivalence relations. II

- For example, whether or not $g, h \in G$ are in the same coset modulo G^{000} depends on their types over M_0. And the equivalence relation on types corresponding to being in the same coset mod G^{000} is K_σ.
- We have confirmed that in the cases above (in Theorems 0.1, 0.2, 0.3), G^{00}/G^{000} is *Borel equivalent* to the appropriate quotient $\widehat{\mathbb{Z}}/\mathbb{Z}$, $SO_2(\mathbb{R})/\langle\alpha\rangle$, or A/Λ.
- These equivalence relations are all Borel equivalent to E_0 (eventual equality on infinite sequences of 0's and 1's.), which is the least "nonsmooth" (or non classifiable) Borel equivalence relations.

Borel equivalence relations. III

- A modification of the example in Theorem 0.1 (namely considering instead of the universal cover of $SL_2(\mathbb{R})$, the product of all the finite covers of $SL_2(\mathbb{R})$) yields a $*$-definable group G (in RCF) such that $G = G^{00}$ and G/G^{000} is (up to Borel equivalence) ℓ^∞, the most complicated K_σ-equivalence relation.

Final remarks and questions

- The quotients we obtained above are among the classical "bad quotients" studied in noncommutative geometry, so it may be fruitful or interesting to take this point of view, rather than that of descriptive set theory.
- Can one find definable G such that G^{00}/G^{000} is not commutative?
- Likewise find examples of first order theories T such that the kernel of $Gal_L(T) \to Gal_c(T)$ is noncommutative.
- Equip covers of surfaces with suitable structure to obtain new examples of non G-compact theories.