Some
Unlikely Intersections
Beyond
André-Oort

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Recent Developments in Model theory

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I.

Diophantine geometry **in** o-minimal structures

Result (+Alex Wilkie) about the distribution of rational points on a "definable set".

II.

Diophantine geometry **via** o-minimal structures

A strategy proposed by Umberto Zannier in the context of the **Manin-Mumford conjecture (Raynaud's Thm)**.

Some cases of the **André-Oort conjecture**, some cases of the **Zilber-Pink conjecture**.

+ Zannier, Masser-Zannier, JP, + Habegger, +Tsimerman, others.

Various uses of o-minimality.

I.

Height of rational points

$$H(a/b) = \max(|a|, |b|), \quad (a,b) = 1,$$

$$H(q_1, \ldots, q_n) = \max(H(q_1), \ldots, H(q_n)).$$

Definition. The **algebraic part** of $Z \subset \mathbb{R}^n$ is

$$\mathrm{Alg}(Z) = \bigcup A$$

over all connected positive dimensional semi-algebraic $A \subset Z$.

Here: a **semi-algebraic set** in $\mathbb{R}^n$ is a finite union of sets, each defined by equations

$$F_i(x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, k,$$

$$G_j(x_1, \ldots, x_n) > 0, \quad j = 1, \ldots, h$$

where $F_i, G_j \in \mathbb{R}[X_1, \ldots, X_n]$.

Counting rational points

Idea: A "reasonable" set $Z \subset \mathbb{R}^n$ has "few" rational points outside its algebraic subset:

Theorem. (+Alex Wilkie) *Let $Z \subset \mathbb{R}^n$ be a set that is definable in an o-minimal structure over $\mathbb{R}$, and $\epsilon > 0$. Then*

$$N(Z - \mathsf{Alg}(Z), T) \leq c(Z, \epsilon) T^{\epsilon}.$$

The "algebraic subset" $\mathsf{Alg}(Z)$ of a set can be viewed as a (weak) analogue of $\mathsf{Sp}(V)$.

Refinement. The same for algebraic points of some bounded degree k:

$$Z \subset \mathbb{R}^n, \quad N_k(Z, T) =$$

$$\#\{(x_1, \ldots, x_n) \in Z : [\mathbb{Q}(x_i) : \mathbb{Q}] \leq k, H(x_i) \leq T\},$$

$$N_k(Z - \mathsf{Alg}(Z), T) \leq c(Z, k, \epsilon) T^{\epsilon}.$$

Further refinement

The theorem yields more information about how much of Alg(Z) we need to remove:

Theorem. Let $Z \subset \mathbb{R}^n$ be definable, $\epsilon > 0$. Then $Z(\mathbb{Q}, T)$ is contained in at most $c(Z, \epsilon)T^\epsilon$ **blocks** coming from finitely many (depending on ϵ) block families.

Definition. A **block** is a cell that is contained in a semi-algebraic cell of same dimension.

* a block of dimension 0 is a point

* a block of positive dimension $\subset$ Alg(Z)

* $Z(k, T)$ in $c(Z, k, \epsilon)T^\epsilon$ blocks.

In general, this result cannot be much improved.

In particular, examples (in $\mathbb{R}_{\mathrm{an}}$) show that one cannot replace $c(Z,\epsilon)T^{\epsilon}$ by

$$c(Z)(\log T)^{C}.$$

Wilkie's conjecture. For $Z \subset \mathbb{R}^n$ definable in $\mathbb{R}_{\mathrm{exp}}$ one can.

Partial results:

Curves (Butler, Jones-Thomas (+Miller))

Certain surfaces (Butler, Jones-Thomas)

II.

Umberto Zannier proposed: strategy for a new proof of Manin-Mumford conjecture (Raynaud's theorem) for abelian varieties $A/\overline{\mathbb{Q}}$.

Same strategy has wider applicability.

Sketch first for multiplicative MM (**torsion** case of theorem of M. Laurent).

1. The multiplicative MM

Algebraic subvariety $V \subset (\mathbb{C}^*)^n$:

$$V = \{\mathbf{x} \in (\mathbb{C}^*)^n : F_i(\mathbf{x}) = 0, i = 1, \ldots, m\}$$

where $\mathbb{C}^* = \mathbb{C} - \{0\}$ as multiplicative group (coordinate-wise multiplication on $(\mathbb{C}^*)^n$).

Consider: **torsion points** on V = points whose coordinates are roots of unity.

"Conjecture": V contains only **finitely many** torsion points **unless** V contains a subtorus of positive dimension or translate thereof by a torsion point ("torsion coset").

Subtorus: equations like: $x^2y^3z = 1$ in $(\mathbb{C}^*)^3$.

Torsion coset: eqs like: $x^2y^3z = \exp(2\pi i/7)$.

"Conjecture": $V \subset (\mathbb{C}^*)^n$ contains only finitely many torsion points **unless** V contains a torus coset of positive dimension.

Observe:

1. Torsion cosets of positive dimension contain infinitely many rational points

2. A torsion point is a torsion coset of the trivial subgroup of $(\mathbb{C}^*)^n$

"Refined conjecture": Finitely many torsion cosets contained in V contain all the torsion points in V. I.e. V has only finitely many **maximal** torsion cosets.

Proof. Since torsion points are algebraic, we can assume V is defined over a number field.

Start with **uniformisation**

$$\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n,$$

$$\exp(z_1, \ldots, z_n) = (\exp(z_1), \ldots, \exp(z_n)).$$

Real coordinates on $\mathbb{C}^n$: $\mathrm{Re}(z), \mathrm{Im}(z)/2\pi$. Then pre-images of torsion points

$$(..., q_j \pi i, \ldots), \quad q_j \in \mathbb{Q}$$

are **rational points**. The uniformization is

$$2\pi i\mathbb{Z}-\text{periodic},$$

so cannot be **definable**. But its **restriction to a fundamental domain** F **is definable** in $\mathbb{R}_{\text{an, exp}}$ (need exp on $\mathbb{R}$ and $\sin, \cos$ on $[0, 2\pi]$).

Let

$$Z = \exp^{-1}(V) \cap F.$$

Opposing bounds

Count rational points in $Z = \mathsf{exp}^{-1}(V) \cap F$.

Archimedean upper bound for Z by PW:

$$N(Z - \mathsf{Alg}(Z), T) \leq c(Z, \epsilon)T^{\epsilon}.$$

Galois lower bound on V side. A torsion point P of order T in $(\mathbb{C}^*)^n$ has degree

$$\phi(T) >> T/\log T,$$

(Euler ϕ-function). A fixed positive proportion of conjugates lie again on V; so if $P \in V$ then

$$N(Z, T) \geq c(V)T/\log T$$

Incompatible bounds: take $\epsilon = 1/2$ (say).

So either the orders of torsion points on V are bounded, giving finiteness, or $\mathsf{Alg}(Z) \neq \emptyset$.

The algebraic part

Next: characterise Alg(Z). Real $\to$ complex.

$$\text{Alg}(\exp^{-1}(V)) = \bigcup \text{complex algebraic } W$$

Let W irreducible complex algebraic variety with

$$W \subset \exp^{-1}(V) \subset \mathbb{C}^n$$

(won't be contained in Z). Let

$$\overline{z_i} \in \mathbb{C}(W)$$

be induced by the coordinate functions, then

$$\exp(\overline{z_i})$$

as functions on W satisfy the equations of V:
Dependent exponentials of algebraic fns.

Ax (1971): Proved Schanuel conjecture in a differential field (i.e. for functions).

By **"Ax-Lindemann-Weierstrass"** = part of Ax-Schanuel corresponding to LW, the $\overline{z_i}$ are linearly dependent over $\mathbb{Q}$ modulo constants.

Ax-Lindemann-Weierstrass

Ax-L-W theorem: *Suppose $a_i \in \mathbb{C}(W)$ are elements in some algebraic function field. The functions*

$$\exp(a_i)$$

on W are **algebraically independent** *over $\mathbb{C}$* **unless** *the a_i are linearly dependent over $\mathbb{Q}$ modulo constants (i.e. $\sum q_i a_i = c \in \mathbb{C}, q_i \in \mathbb{Q}$, not all =0) .*

is **equivalent** (more generally) to:

Theorem ("Ax-L-W"): *Let $V \subset (\mathbb{C}^*)^n$ be algebraic. A maximal complex algebraic variety $W \subset \exp^{-1}(V)$ is a translate of a rational linear subspace.*

Conclude:

$\text{Alg}(\exp^{-1}(V)) = \bigcup \exp^{-1}$ subtorus cosets in V

(not only *torsion* cosets).

Summary/conclusion

Transcendental uniformization, definable on a fundamental domain:

rational point $\leftrightarrow$ torsion point

"Complexity" (order) of torsion point:

upper bound $<<$ lower bound

Characterization of "algebraic part" (Ax-L-W):

maximal algebraic $\approx$ subtorus coset

Finiteness for the number of subtori T having a coset $aT \subset V$ (elementary/o-minimality).

Finally: an inductive argument to conclude:

tor csts $aT \subset V \leftrightarrow$ tor pts $a \in V' \subset (\mathbb{C}^*)/T$.

Completes proof .□

2. Andre-Oort Conjecture

André-Oort conjecture (‘89/‘95): analogue of MM for **Shimura varieties** X. Examples:

* Moduli space of pp abelian vars given dim
* Hilbert modular surfaces, H modular varieties
* Shimura curves: quotient of $\mathbb{H}$ by a discrete subgroup of $SL_2(\mathbb{R})$ coming from an indefinite quaternion algebra over $\mathbb{Q}$, gen modular curves.

Conjecture. Let $V \subset X$. Then V contains only finitely many **“special points”** unless it contains a **“special subvariety”** of pos. dim.

So: “special pt” $\sim$ torsion pt, “sp subv.” $\sim$ ...

Refined version: All “special points” $\in V$ lie in finitely many “special subvarieties” $\subset V$.

Full proof announced by Klingler-Ullmo-Yafaev on GRH. Few cases known unconditionally.

André-Oort for $\mathbb{C}^n$

$\mathbb{C} = Y(1)$ as j-line parameterising elliptic curves.

$j(\tau)$: j-invariant of $E \leftrightarrow \mathbb{Z} \oplus \mathbb{Z}\tau$, $\mathrm{SL}_2(\mathbb{Z})$-inv.

Special point in $\mathbb{C}$ = the j invariant of a CM elliptic curve = elliptic curve with extra endomorphisms. **Special point** in $\mathbb{C}^n$: tuple.

André-Oort Conjecture for $\mathbb{C}^n$: $V \subset \mathbb{C}^n$ has finitely many special points **unless** it contains a **"special subvariety"** of positive dimension ($\approx$ product of modular curves).

Edixhoven (2005) **under GRH for CM fields**. For $n = 2$, André **unconditionally** (1998).

Sketch proof.

Reprise mult MM proof with j instead of exp.

Uniformisation:

$$j : \mathbb{H}^n \to \mathbb{C}^n,$$

$$j(\tau_1, \ldots, \tau_n) = (j(\tau_1), \ldots, j(\tau_n)).$$

$$\mathrm{SL}_2(\mathbb{Z})^n - \text{invariant}, \quad \tau \mapsto \frac{a\tau + b}{c\tau + d}$$

Definability of j **on** F, in $\mathbb{R}_{\text{an exp}}$, despite its essential singularity in cusp, by q-expansion, or Peterzil+Starchenko ('04) result for $\wp(z, \tau)$. So too j on F^n.

$j(\tau)$ is special $\iff$ τ is **imaginary quadratic.**

By Complex Multiplication

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(D)$$

Opposing bounds

Definability + bounded degree: **Upper bound.**

$$N_2(Z - \mathrm{Alg}(Z), T) \leq c(Z, \epsilon)T^{\epsilon}.$$

Lower bound: $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = h(D)$. Siegel:

$$h(D) \geq c(\eta)|D|^{1/2-\eta}, \quad \eta > 0,$$

unconditional (though ineffective). And as $H(\tau) << D$, if $j(\tau_1, \ldots, \tau_n) \in V$, $D_i = D(\tau)$ and $D = \max D_i$ get

$$N_2(Z) \geq c(V)D^{1/4} \quad (\eta = 1/4 \text{ say }).$$

Incompatible bounds.

Study Alg(Z). Last ingredient:

Ax-Lindemann-Weierstrass for j

If $g \in \mathrm{GL}_2^+(\mathbb{Q})$ ($+$ for det > 0, to preserve $\mathbb{H}$),

$$j(\tau), \quad j(g\tau), \quad g\tau = \frac{a\tau + b}{c\tau + d}$$

are related by a **modular polynomial**,

$$\Phi_N(j(\tau), j(g\tau)) = 0,$$

and so are algebraically dependent (over $\mathbb{Q}$).

Definition. Algebraic functions $a_i \in \mathbb{C}(W)$ will be called **geodesically independent** if they are all non-constant and there are no relations $a_i = g a_j, i \neq j$ as above.

Need: all a_i take values in $\mathbb{H}$ for some point of W so that $j(a_i)$ are locally functions on W.

Theorem (Ax-L-W for j): *Suppose a_i are geodesically independent algebraic functions. Then $j(a_i)$ are algebraically independent $/\mathbb{C}$.*

Definition. A **weakly special** subvariety of $\mathbb{H}^n$ is $W \cap \mathbb{H}^n$ where W is defined by equations

$$z_{i_k} = g_k z_{j_k}, \quad g_k \in \mathrm{GL}_2^+(\mathbb{Q}), \quad k = 1, \ldots, \ell$$

$$z_{\ell_k} = c_k \in \mathbb{H}, \quad k = 1, \ldots, m.$$

It is **special** if all c_k are quadratic.

This data determines a special subvariety

$$W_{\{(i_k, j_k, g_k)\}}$$

on the variables i_k, j_k.

We refer to W as being the **translate** by the c_k of $W_{\{(i_k, j_k, g_k)\}}$.

Theorem (Ax-L-W for j). *Let $V \subset \mathbb{C}^n$ be algebraic. If W is a maximal complex algebraic variety with $W \cap \mathbb{H}^n \subset j^{-1}(V)$ then W is weakly special.*

Proof. Uses O-minimality plus P-Wilkie again.

Ingredients and prospects for AO

Basic set up: Uniformisation

$$\pi : U \to X, \quad \Gamma - \text{invariant}.$$

All cases of (mixed) André-Oort look like this. Special points in U have finite degree..

A. Definability (upper bounds): Definability of uniformising map. Peterzil-Starchenko: for theta functions in both sets of variables (in $\mathbb{R}_{\text{an,exp}}$), so for $\mathcal{A}_{g,1}$, even as mixed Shimura variety.

B. Lower bounds: for Galois orbits of special points: Jacob Tsimerman (2011): $\mathcal{A}_g, g \leq 5$ unconditionally. (Also Yafaev-Ullmo).
Height of point in F (Tsimerman, for $\mathbb{H}_g$).

C. Ax-Lindemann-Weierstrass: Of interest and approachable indpt of lower bounds.

Maximal algebraic $\subset \pi^{-1}(V)$ *is weakly special.*

Further results

Cases of ZP – later:

1. Masser-Zannier "torsion anomalous" points

2. "unlikely" in $\mathbb{C}^n$ (+Habegger)

Cases of AO or "generalised" versions:

3. AOMML for $\mathbb{C}^n \times E_1 \times \ldots \times E_m \times (\mathbb{C}^*)^\ell$, $E_i/\overline{\mathbb{Q}}$

4. Hilbert modular surfaces (Daw-Yafaev 2011)

In progress:

5. Products of elliptic modular surfaces: L^n,

$$L = \{(\lambda, x, y) : y^2 = x(x-1)(x-\lambda)\}$$

Special point: λ_i special, (x_i, y_i) torsion.

6. Products of Shimura curves

7. Siegel modular threefold $\mathcal{A}_{2,1}$ = moduli space of pp Abelian surfaces: (+ Tsimerman)

3. The Zilber-Pink conjecture

A far-reaching generalization of AOMM, due to Zilber ($(\mathbb{C}^*)^n$, semi-abelian), independently (later) Pink for (mixed) Shimura varieties, also Bombieri-Masser-Zannier proved results, made conjectures on "unlikely intersections" in $(\mathbb{C}^*)^n$.

Let $\mathcal{S}^{[k]}$ be the union of all algebraic subgroups of $(\mathbb{C}^*)^n$ of codimension at least k.

E.g. For a curve $C \subset \mathbb{G}_{\mathrm{m}}^n(\mathbb{C}) = (\mathbb{C}^*)^n$, $C/\mathbb{C}$.

Conjecture. $C \cap \mathcal{S}^{[2]}$ is **finite**, **unless** C is contained in a proper algebraic subgroup.

This is a **Theorem** due to BMZ, Maurin.

$C \cap \mathcal{S}^{[2]}$ consists: $(x_1, \ldots, x_n) \in C$ satisfying 2 (or more) independent multiplicative relations.

Multiplicative MM is a special case ($n = 2$ or intersect with subgroups of codimension n)

Example. Find all $t \in \mathbb{C}$ such that

$$(t, 1+t, 1-t) \in \mathbb{C}^3$$

satisfy **two** independent multiplicative relations (Cohen-Tretkoff+Zannier). Or same for

$$(2, 3, t, 1+t, 1-t) \subset \mathbb{C}^5.$$

ZP implies ML

Suppose all but 2 coordinates constant on C: $C = \{(c_1, \ldots, c_n, x, y) : f(x, y) = 0\}$. (assume: c_i mult. ind. o/w $C \subset$ special). Two equations

$$x^a y^b = c_1^{\alpha_1} \ldots c_n^{\alpha_n}, \quad x^c y^d = c_1^{\beta_1} \ldots c_n^{\beta_n}$$

amounts to: solving $f(x, y) = 0$ in the **division group** generated by $c_1, \ldots, c_n$.

I.e. Although ZP involves only special subvts, Mordell-Lang appears as a degenerate case.

ZP for curves in $\mathbb{C}^n = Y(1)^n$

Conjecture. Let $C/\mathbb{C}$ be a curve in $\mathbb{C}^n$. Then the intersection of C with the **union** $\mathcal{S}^{[2]}$ of all special subvarieties of $\mathbb{C}^n$ of codimension ≥ 2 is **finite** – **unless** C is **contained** in a proper special subvariety of $\mathbb{C}^n$.

Theorem. (+Habegger) *The conjecture above is true if C is defined over $\overline{\mathbb{Q}}$ and* **asymmetric***.*

Definition: C is **asymmetric** if each positive integer appears at most once among $\deg(X_i|C)$, up to one exception which may appear twice.

Same strategy. First "unlikely" result "beyond AO". Requires "Ax-log" result for j. Includes an analogue of ML (holds for all $V \subset \mathbb{C}^n$).

Consider now $C \subset \mathbb{C}^n$ as Shimura variety.

$\mathcal{S}^{[2]} = \cup$ special subvarieties of codimension 2.

$C \cap \mathcal{S}^{[2]}$ consists: $(x_1, \ldots, x_n) \in C$ satisfying 2 independent **modular** relations (or coordinates special).

Suppose all but 2 coordinates constant on C:

$C = \{(c_1, \ldots, c_n, x, y) : f(x, y) = 0\}$

$\Phi_n(x, c_i), \Phi_m(y, c_j)$ (or x and/or $y =$ special).

...analogue of Mordell-Lang for $V \subset Y(1)^n$.

“Mordell-Lang” for $\mathbb{C}^n$

Definition. Let Σ be a finite subset of $\mathbb{C}$. A point $x \in \mathbb{C}$ is called Σ-special if it is special or in the Hecke orbit of some $c \in \Sigma$ i.e. $j^{-1}(x) \in \mathrm{GL}_2^+(\mathbb{Q}) j^{-1}(c)$.

Definition. A Σ-**special subvariety** is a weakly special subvariety which contains a Σ-special point.

Theorem. (+Philipp Habegger) *Let $\Sigma \subset \overline{\mathbb{Q}}$ be a finite set and $V \subset \mathbb{C}^n$ a variety. Then V contains only finitely many Σ-special points* **unless** *V contains a Σ- special subvariety of positive dimension.* **Moreover**, *V contains only finitely many maximal Σ-special subvarieties.*

Sketch. Note that for $c \in \overline{\mathbb{Q}}$ but not special, a point $\sigma \in \mathbb{H}$ with $j(\sigma) = c$ is **transcendental** (Th. Schneider).

Fixing one such $\sigma \in \mathbb{H}$, the Hecke orbit is

$$\mathrm{GL}_2^+(\mathbb{Q})\sigma = \{g\sigma : g \in \mathrm{GL}_2^+(\mathbb{Q})\}.$$

Take $\sigma \in \mathbb{H}$ with $j(\sigma) = c$ for each $c \in \Sigma$.

Break into finitely many cases:

* Certain coords, say $x_{k+1}, \ldots, x_n$ are special.

* Other x_j is in Hecke orbit of some $c_j \in \Sigma$.

For each such case consider:

$$\Omega = \mathrm{GL}_2(\mathbb{R})^k \times \mathbb{H}^{n-k} \to U = \mathbb{H}^n \to \mathbb{C}^n,$$

$$(g_i, \tau_j) \mapsto (g_i\sigma_i, \tau_j) \mapsto (j(g_i\sigma_i), j(\tau_j))$$

and look for suitably "rational" points in the preimage of Z in $\mathrm{GL}_2(\mathbb{R})^n$:

Quadratic points in $\mathbb{H}$, rational points in $\mathrm{GL}_n(\mathbb{R})$.

The map $\mathrm{GL}_n(\mathbb{R}) \to \mathbb{H}$ is fibred by copies of $\mathrm{SO}_2(\mathbb{R}) \times \Delta$.

But we get T^ϵ "blocks". The map $\mathrm{GL}_2(\mathbb{R}) \to \mathbb{H}$ is semialgebraic, so image of block is a finite union of blocks.

So T^ϵ blocks in Ω map to T^ϵ blocks in U.

Alg(Z) is same as before.

Lower bounds:

* Special points: same

* Orbit of c: isogeny estimates (Masser et al) or Serre open image. $\square$

Certain $C \subset \mathbb{C}^n$

Sketch. What does an "unlikely intersection" point look like?

$$(x_1, \ldots, x_n) \in C$$

Cases:

(1) $\Phi_N(x_{i_1}, x_{i_2}) = 0$ and $\Phi_M(x_{i_3}, x_{i_4}) = 0$, with $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}$ distinct.

(2) $\Phi_N(x_{i_1}, x_{i_2}) = 0, \Phi_M(x_{i_2}, x_{i_3}) = 0$, with $x_{i_1}, x_{i_2}, x_{i_3}$ distinct.

(3) $x_{i_1} = c$ special, $\Phi_M(x_{i_2}, x_{i_3}) = 0$, with $x_{i_1}, x_{i_2}, x_{i_3}$ distinct.

(4) $x_{i_1} = c_1, x_{i_2} = c_2$ with x_{i_1}, x_{i_2} distinct and c_1, c_2 special reverts to AO.

Case (2) on x_1, x_2, x_3

Consider: Points $P = (x_1, x_2, x_3) \in C$ with $\Phi_N(x_1, x_2) = 0$, and $\Phi_M(x_2, x_3) = 0$, where N, M depend on P.

Uniformisation: $\mathbb{H}^3 \to \mathbb{C}^3$ by j-function.

P as above gives rise to $(\tau_1, \tau_2, \tau_3) \in \mathbb{H}^3$ with

$$z_2 = \alpha z_1, \quad z_3 = \beta z_2$$

for some $\alpha, \beta \in \mathrm{GL}_2^+(\mathbb{Q})$.

If some coordinate is constant on C we are in "Mordell-Lang" situation: we may assume C is not contained in any weakly special subvariety.

Need suitable "Ax-type" result:

“Ax logarithms”

$j : \mathbb{H} \to \mathbb{C}$ has a multivalued inverse $\ell : \mathbb{C} \to \mathbb{H}$, the “$j$-logarithm”.

Want: for algebraic functions a_i, the $\ell(a_i)$ are algebraically independent unless the a_i have modular relations, or are constant.

Theorem. *Let $C \subset \mathbb{C}^3$ be irreducible curve, $\tau \in j^{-1}(C) \subset \mathbb{H}^3$. Suppose a complex algebraic hypersurface W contains a neighbourhood of z in $j^{-1}(C)$. Then C is contained in a weakly special subvariety.*

Uses: André’s normality theorem (does not use o-minimality).

Case (2), ctd

For $\alpha, \beta \in \mathsf{GL}_2^+(\mathbb{R})$, let

$$Y_{\alpha,\beta} = \{(\tau_1, \tau_2, \tau_3) \in \mathbb{C}^3 : \tau_2 = \alpha\tau_1, \tau_3 = \beta\tau_2\}.$$

$Y_{\alpha,\beta}$ is a complex algebraic curve in a family parameterised by

$$\mathsf{GL}_2^+(\mathbb{R}) \times \mathsf{GL}_2^+(\mathbb{R}).$$

Let $Z = j^{-1}(C) \cap F$, definable. Also definable:

$$X = \{(\alpha, \beta) \in \mathsf{GL}_2^+(\mathbb{R})^2 : Y_{\alpha,\beta} \cap Z \neq \emptyset\}.$$

1. Each $Y_{\alpha,\beta} \cap Z$ is finite, otherwise, by "Ax-log", C would be contained in a weakly special subvariety, contrary to assumptions.

2. O-minimality: a uniform finite bound for $(\alpha, \beta) \in \mathsf{GL}_2^+(\mathbb{R})^2$.

3. The intersections are then given by finitely many functions f_i defined and C^1 on some cells.

Case (2), concluded

4. Lower bounds: an unlikely point P has "many" Galois conjugates: i.e. gives rise to at least cT^δ points of height $\leq T$ on X.

Uses: height properties on curves (asymmetry used here), isogeny estimates, ...

5. Choose $\epsilon < \delta$. Pila-Wilkie now provides a finite number of definable "block families" containing all the "blocks" occurring in the theorem, compatible with the cells for the f_i.

6. Suppose now a point P with $L = \max(N, M)$ large. Have $\geq cL^\delta$ points in Z. But the points $Q \in X$ lie on $\leq CL^\epsilon$ blocks. If an algebraic curve through a point Q has f_i non-constant, we get an algebraic surface W containing Z. **Contradiction.** So the f_i are all constant on the blocks, and this accounts for too few points P. **Contradiction.** □

“Torsion anomalous” points

Masser-Zannier establish first cases of Pink’s relative MM conjecture, using o-minimality.

Theorem. (M+Z) *There are only finitely many complex numbers* $\lambda \neq 0, 1$ *such that the points*

$$(2, \sqrt{2(2-\lambda)}), \quad (3, \sqrt{6(3-\lambda)})$$

on the elliptic curve

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

are **both** *torsion points.*

View as family of $E_\lambda \times E_\lambda$ over λ-line. The point $((2, ...), (3, ...))$ describes a curve, on which one expects only finitely many torsion points. But the ambient abelian variety moves with λ.

For $(2, \sqrt{2(2-\lambda)})$ alone, infinitely many λ.

4. André-Oort again

Ingredients

A. Definability: Peterzil-Starchenko $\mathcal{A}_{g,1}$.

B. Lower bounds: Tsimerman $\mathcal{A}_{g,1}, g \leq 5$.

C. Ax-Lindemann-Weierstrass:

Consider Shimura variety X e.g. $\mathcal{A}_{g,1}$. Have

$$\pi : U \to X, \quad V \subset X$$

Conjecture. (Ax-L-W): A maximal complex algebraic $W \cap U \subset \pi^{-1}(V)$ is weakly special.

Theorem. (Ullmo-Yafaev) *True if* $\dim V = 1$.

Hilbert modular surfaces

Certain quotient of $\mathbb{H}^2$ by action of a discrete arithmetic group coming from a real quadratic field k.

$$\pi : \mathbb{H}^2 \to X.$$

Moduli space of pp Abelian surfaces with real multiplication: $X \subset \mathcal{A}_{2,1}$.

Theorem. (Daw-Yafaev) *AO for HMS's*

Definability: Peterzil-Starchenko for $\mathcal{A}_{2,1}$. Lower bounds: Edixhoven. AxLW: Ullmo-Yafaev. □

Other cases of curve V in $X \subset \mathcal{A}_{g,1}, g \leq 5$ should follow similar lines.

Siegel modular threefold

AO for moduli space of pp Abelian surfaces, + Jacob Tsimerman.

Siegel upper half space: $J : \mathbb{H}_2 \to \mathcal{A}_{2,1}$

Definability: Peterzil-Starchenko.

Lower bound for Galois orbit: Tsimerman.

Ax-Lindemann-Weierstrass: uses o-minimality, but not P-Wilkie.

Take $V \subset \mathcal{A}_{2,1}$

* $\dim V = 1$: conclude using Ullmo-Yafaev.

* $\dim V = 2$: ... tame complex analytic results of Peterzil-Starchenko ...