# Homogeneity of the free group

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# Introduction

Denote by $\mathbb{F}_k$ the free group on $a_1, \ldots, a_k$.
**Tarski problem (1945):** Are $\mathbb{F}_k$ and $\mathbb{F}_n$ elementary equivalent for $k \neq n$?

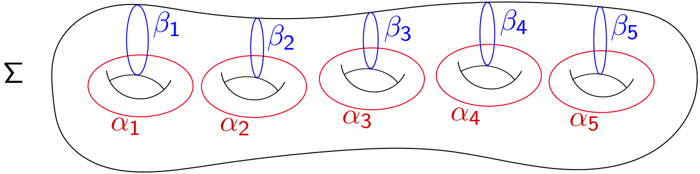
**Theorem (Kharlampovich-Myasnikov, Sela)**

The canonical embedding of $\mathbb{F}_k$ in $\mathbb{F}_m$ is elementary for all $m \geq k \geq 2$.

Thus all free groups of rank $\geq 2$ are elementary equivalent.
The techniques used by Sela are mostly geometric.

**Theorem (Kharlampovich-Myasnikov, Sela)**

Let $\Sigma$ be a closed surface with $\chi(\Sigma) < -1$.
Then $\mathrm{Th}(\pi_1(\Sigma)) = \mathrm{Th}(\mathbb{F}_2)$.




$$\pi_1(\Sigma) = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \mid [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = 1 \rangle$$

+ Complete description of finitely generated models of the theory of free groups.

# Introduction

Using the tools he developed for Tarski problem, Sela shows:

**Theorem (Sela)**

The theory of the free group $\mathrm{Th}(\mathbb{F}_k)$ is stable.

It was known (Poizat) that the free group $\mathbb{F}_\omega$ is connected.
$\Rightarrow$ it has a unique generic type $p_0$.

**Theorem (Pillay)**

The elements realizing the generic type $p_0$ in the free group $\mathbb{F}_k$ are exactly the primitive elements.

(An element in a free group is said to be primitive if it is part of a basis.)

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**Consequence**

If an element $u$ of $\mathbb{F}_k$ has the same type as $a_1$, then $u$ is primitive, in particular there is an automorphism $\sigma$ of $\mathbb{F}_k$ with $\sigma(u) = a_1$.

A countable group $G$ is **homogeneous** if for all $l \in \mathbb{N}$,

$$\mathrm{tp}^G(g_1, \ldots, g_l) = \mathrm{tp}^G(g'_1, \ldots, g'_l)$$

$\iff$ there is $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(g_i) = g'_i$ for $1 \leq i \leq l$.

**Theorem (P.-Sklinos, Ould Houcine)**

The free group $\mathbb{F}_k$ is homogeneous.

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**Theorem (P.-Sklinos)**

The fundamental group $\pi_1(\Sigma)$ of a closed surface $\Sigma$ of characteristic at most $-3$ is not homogeneous.

# Homogeneity of the free groups: some idea of the proof

$\mathbb{F}_k = \langle a_1, \ldots, a_k \rangle$. Let $u, v \in \mathbb{F}_k$ such that $\mathrm{tp}^{\mathbb{F}_k}(u) = \mathrm{tp}^{\mathbb{F}_k}(v)$.

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$$\begin{aligned} \theta(u) &= \theta(w_u(a_1, \dots, a_k)) \\ &= w_u(\theta(a_1), \dots, \theta(a_k)) \\ &= w_u(b_1, \dots, b_k) = v. \end{aligned}$$

$\mathbb{F}_k = \langle a_1, \dots, a_k \rangle$. Let $u, v \in \mathbb{F}_k$ such that $\mathrm{tp}^{\mathbb{F}_k}(u) = \mathrm{tp}^{\mathbb{F}_k}(v)$. Assume moreover that neither $u$ nor $v$ contained in a proper free factor of $\mathbb{F}_k$.

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An injective morphism $\mathbb{F}_k \to \mathbb{F}_k$ which fixes $u$ is also surjective.

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If $\theta, \theta' : \mathbb{F}_k \to \mathbb{F}_k$ injective with $\theta(u) = v$ and $\theta'(v) = u$, the homomorphism $\theta' \circ \theta$ is injective and fixes $u \Rightarrow$ it is surjective, hence so is $\theta$.

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Take $b_1, b_2$ solution, and $\theta$ defined by $\theta(a_j) = b_j$, then $\theta(u) = v$.
$\theta(\mathbb{F}_2) = \langle b_1, b_2 \rangle$ is free of rank 2.
$\mathbb{F}_2 \overset{\theta}{\twoheadrightarrow} \theta(\mathbb{F}_2) \simeq \mathbb{F}_2$ but free groups are Hopfian so $\theta$ is injective.

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Equivalently, if we define

$$\eta : \mathbb{F}_2 \twoheadrightarrow \mathbb{F}_2/\langle\langle[a_1, a_2]\rangle\rangle$$

then $\theta : \mathbb{F}_2 \to \mathbb{F}_2$ is injective $\iff$ it does not factor through $\eta$.

# Homogeneity of the free groups: some idea of the proof

In general case, we will use

**Theorem**

$u, v \in \mathbb{F}_k$ and $u$ is not contained in a proper free factor of $\mathbb{F}_k$. There exists a finite set of proper quotients $\eta_j : \mathbb{F}_k \twoheadrightarrow Q_j$ such that any homomorphism $\theta : \mathbb{F}_k \to \mathbb{F}_k$ such that $\theta(u) = v$ which is not injective factors through one of the quotients $\eta_j$ **after precomposition by an element** $\sigma$ **of** $\mathrm{Aut}_{\langle u \rangle}(\mathbb{F}_k)$

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**Problem:** Need now to express precomposition by an automorphism fixing $u$.
**Idea:** Use JSJ decomposition of $\mathbb{F}_k$ with respect to $u$.

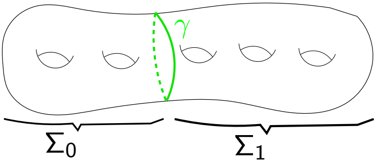
To show nonhomogeneity of surface groups, need to know something about elementary submodels of surface groups.

**Theorem (Kharlampovich-Myasnikov, Sela)**

The canonical embedding $\mathbb{F}_m \hookrightarrow \mathbb{F}_n$ for $n \geq m \geq 2$ is elementary.

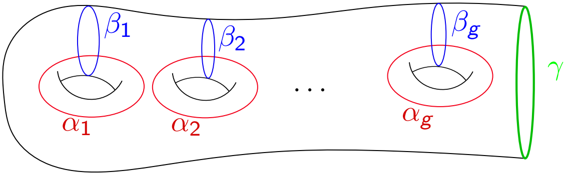
**Question**

Elementary subgroups of surface groups?




Is $\pi_1(\Sigma_0)$ elementary in $\pi_1(\Sigma)$? Is $\pi_1(\Sigma_1)$?

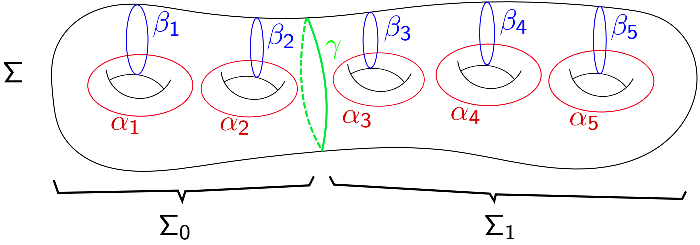
**Fact:** Consider the following surface $\Sigma_0$ with one boundary component




$\pi_1(\Sigma_0)$ is a free group on $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$, and

$$\gamma = \Pi_{i=1}^{g}[\alpha_i, \beta_i]$$

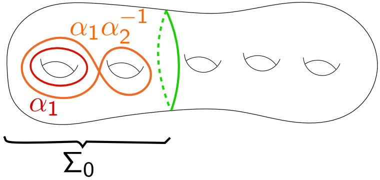
# Elementary embeddings




$\exists xyx'y' \;\; \gamma = [x, y][x', y']$ is true on $\pi_1(\Sigma)$ but not on $\pi_1(\Sigma_1)$.

**Fact:** $\pi_1(\Sigma_0)$ is elementary in $\pi_1(\Sigma)$.

# Proof of non-homogeneity of surface groups




- Here are two bases for $\pi_1(\Sigma_0) \simeq \mathbb{F}_4$: $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ and $\{\alpha_1\alpha_2^{-1}, \beta_1, \alpha_2, \beta_2\}$
- Both $\alpha_1$ and $\alpha_1\alpha_2^{-1}$ primitive in $\pi_1(\Sigma_0)$ so they have the same type in $\pi_1(\Sigma_0)$.
- $\pi_1(\Sigma_0)$ elementary so they have the same type in $\pi_1(\Sigma)$
- But no automorphism of $\pi_1(\Sigma)$ sends $\alpha_1$ to $\alpha_1\alpha_2^{-1}$.