Cartan subgroups of definable groups

Margarita Otero
Universidad Autónoma de Madrid
(joint work with Elías Baro and Eric Jaligot)

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Cartan subgroups

Let G be a group. A subgroup Q of G is a Cartan subgroup of G if

- Q is a maximal nilpotent subgroup of G; and
- for every $X \trianglelefteq Q$, $|Q : X|$ finite implies $|N_G(X) : X|$ finite.

Carter subgroups

Let G be a group definable in a structure $\mathcal{M}$. A subgroup Q of G is a Carter subgroup of G if

- Q is a definable definably connected nilpotent subgroup of G; and
- $|N_G(Q) : Q|$ is finite (*i.e.*, Q is *almost selfnormalizing*).

Examples

- Cartan: Q maximal nilpotent and $|N_G(X) : X|$ finite, for every $X \trianglelefteq Q$ of finite index.
- Carter: Q nilpotent, definably connected and almost selfnormalizing.

1. Let G be a connected compact Lie group. The maximal tori of G are Cartan subgroups of G. They are also Carter subgroups of G, considering G as a semialgebraic group.
2. Let G be a definably connected definably compact group definable in an o-minimal structure. Let T be a maximal definable-torus of G (*i.e.*, a maximal definably connected abelian subgroup of G). Then T is a Cartan and Carter subgroup of G.

Examples

$$SL_2(\mathbb{R}) := \{A \in GL_2(\mathbb{R}) \mid det(A) = 1\}$$

Cartan subgroups of $SL_2(\mathbb{R})$:

- $Q_1 := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R} \right\}$
- $Q_2 := SO(2, \mathbb{R})$

Carter subgroups of $SL_2(\mathbb{R})$:

- $Q_1^o = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}, a > 0 \right\}$
- $SO(2, \mathbb{R})$

Definition

We say that a group G satisfies the weak hypothesis if G is definable in a structure $\mathcal{M}$ and

1. $\mathcal{M}$ is equipped with a dimension for definable sets which is definable, additive, monotone and such that the finite sets are exactly the definable sets of dimension 0;
2. G has the *dcc* for definable subgroups, and
3. quotients of G by definable equivalence relations are definable.

Fact.

A group definable in an o-minimal structure satisfies the weak hypothesis:
(1) [Pillay 88],
(2) [Pillay 88], and
(3) [Edmundo 03].

Basic lemma to study Cartan subgroups:

Normalizer Condition Lemma

Let G be a nilpotent group satisfying the *weak hypothesis*. Then,

$$|G : H| \text{ infinite implies } |N_G(H) : H| \text{ infinite,}$$

for every $H \leq G$ definable.

Proof. By induction on the nilpotency class of G (we actually only need *dcc*). □

Corollary (Cartan *vs.* Carter)

Let G be a group satisfying the *weak hypothesis*.

- If Q is a Cartan subgroup of G then Q is definable and Q^o is a Carter subgroup of G.
- If Q Carter subgroup of G then there is a Cartan subgroup $\widetilde{Q}$ of G ($\widetilde{Q} \leq N_G(Q)$) such that $\widetilde{Q}^o = Q$.

- Cartan: Q maximal nilpotent and $|N_G(X) : X|$ finite, for every $X \trianglelefteq Q$ of finite index.
- Carter: Q nilpotent, definably connected and almost selfnormalizing.

Remark.
∘ A Carter subgroup is a maximal definably connected nilpotent subgroup.
∘ A selfnormalizing Carter subgroup is also a Cartan subgroup.
∘ A definably connected Cartan subgroup is also a Carter subgroup.

Conjugates

- Fact[Berarducci 08/Edmundo 05].
  Let G be a definably connected definably compact group definable in an o-minimal structure. Then, there is a unique maximal definable-torus, up to conjugacy. Moreover,
  $$T^G := \bigcup_{g \in G} T^g = G.$$

- In $SL_2(\mathbb{R})$ the two examples of Cartan subgroups Q_1:= diagonal matrices and $Q_2 := SO(2, \mathbb{R})$ are not conjugate. But they are the only two Cartan subgroups, up to conjugacy. Its conjugates:
  $$Q_1^{SL_2(\mathbb{R})} = \{A \in SL_2(\mathbb{R}) : |\mathrm{tr}(A)| > 2\} \cup \{I, -I\}$$
  $$Q_2^{SL_2(\mathbb{R})} = \{A \in SL_2(\mathbb{R}) : |\mathrm{tr}(A)| < 2\} \cup \{I, -I\}$$

For groups definable in o-minimal structure we would like Cartan subgroups play the role that maximal definable-tori play in definably compact groups.

We face with the following problems:

- They may not be definably connected.
- There can be more that one conjugacy class.
- The conjugates of a Cartan subgroup (or even of the union of all Cartan subgroups) may not cover the group:

$$Q_1^{SL_2(\mathbb{R})} \cup Q_2^{SL_2(\mathbb{R})} = \{A \in SL_2(\mathbb{R}) : |\mathrm{tr}(A)| \neq 2\} \cup \{\pm I\} \subset SL_2(\mathbb{R}).$$

Questions

Let G be a d. connected group definable in an o-minimal structure.

- Does G have Cartan subgroups?
- Under what conditions on G, are the Cartan subgroups definably connected?
- Are there finitely many conjugacy classes of Cartan subgroups?
- What does the conjugates of a Cartan subgroup cover?
- What does the union of all Cartan subgroups of G cover?

Lie groups

Fact [Pillay 88].
Any group definable over an o-minimal expansion of the real line is a Lie group.

Fact on Lie groups [*Classical*-Neeb 96].
Let G be a connected Lie group. Then,

1. there exist Cartan subgroups of G;
2. all the Cartan subgroups of G are connected $\iff$ the image of the exponential map is dense in G;
3. there are finitely many conjugacy classes of Cartan subgroups;
4. the union of all (conjugate of) Cartan subgroups of G is dense in G, and
5. if Q Cartan subgroup of G then $\dim((Q^o)^G) = \dim(G)$.

Generosity

Let G be a group satisfying the *weak hypothesis*, and X a definable subset of G.

- X is weakly generous in G if $\dim(X^G) = \dim(G)$.
- X is generous in G if X^G is generic in G (*i.e.*, finitely many translates of X^G cover G).
- X is largely generous in G if X^G is large in G (*i.e.*, $\dim(G \setminus X^G) < \dim(G)$).

$SL_2(\mathbb{R})$

- The Cartan subgroup of diagonal matrices is generous (but not largely generous).
- The Cartan subgroup $SO(2, \mathbb{R})$ is weakly generous (but not generous).

Theorem 1

Let G be a group satisfying the *weak hypothesis*.

- G has at most one conjugacy class of largely generous Carter subgroups.
- If Q is a largely generous Carter subgroup of G then the set $\{x \in G \mid x \text{ is in a unique conjugate of } Q\}$ is large in G.

Notation.

Let G be a group satisfying the *weak hypothesis*. Let X be a definable subset of G.

- $X_r := \{x \in X \mid \dim\{g \in G/N_G(X) \mid x \in X^g\} = r\}$
- $X_0 = \{x \in X \mid x \in \text{only finite many conjugates of } X\}$.
- $X_0^G = \{x \in X^G \mid x \in \text{only finite many conjugates of } X\}$.

Weakly Generousity Lemma

Let G be a group satisfying the *weak hypothesis*. Let $H \leq G$ be definable and W a finite subset of $N_G(H)$. Then

$$\dim(WH)^G = \dim(G) \iff \dim(WH)_0 = \dim(N_G(WH)).$$

In this case

- $(WH)_0^G$ is large in $(WH)^G$, and
- $\dim(WH)_0 = \dim(WH) = \dim(H) = \dim(N_G(H))$.

Proof. Based on the following Fact[Jaligot 06] (taking $X := WH$).
Let G be a group satisfying the *weak hypothesis*. Let X be a definable subset of G. Then, for every $r \leq \dim(G/N_G(X))$ such that $X_r \neq \emptyset$ we have

$$\dim(X_r^G) = \dim(G) + \dim(X_r) - \dim(N_G(X)) - r.$$

□

Remark.

Let $\mathcal{M}$ be an o-minimal structure. Let G be a definably connected definably compact group definable in $\mathcal{M}$.
A maximal definable-torus of G is a Cartan and Carter subgroup.

Proof. Let T be a definable-torus of G. $T^G = G$ implies that $\dim(T) = \dim(N_G(T))$ by the Weakly generousity lemma.Hence, T is a Carter subgroup of G. T being selfcentralizing is also a Cartan subgroup of G. □

Theorem 1

Let G be a group satisfying the *weak hypothesis*.

1. G has at most one conjugacy class of largely generous Carter subgroups.
2. If Q is a largely generous Carter subgroup of G then the set $\{x \in G \mid x \text{ is in a unique conjugate of } Q\}$ is large in G.

Proof.

1. Let $P, Q \leq G$ be largely generous Carter subgroups. P^G and Q^G large in G imply (by the Weakly generousity lemma) that $P_0^G \cap Q_0^G \neq \emptyset$. After conjugation WMA $P_0 \cap Q_0 \neq \emptyset$. Then you get $P = Q$ applying just that definably connected groups act trivially on finite sets, and the normalizer condition.
2. We first apply again the Weakly generousity lemma to get Q_0^G large in G, and then by the same argument as above we get that the elements in Q_0^G are in a unique conjugate of Q. □

From now on, definable means definable in an o-minimal structure.

Theorem 2: Solvable case

Let G be a definable definably connected solvable. Then,

1. there exist Cartan subgroups of G;
2. Cartan subgroups of G are definably connected and selfnormalizing;
3. for any Cartan subgroup Q of G, the set of elements of Q belonging to a unique conjugate of Q is largely generous in G, and
4. Cartan subgroups of G are conjugate.

Proof of theorem 2: Solvable case

Fact 1.
Let G be a definable definably connected solvable group. Then,
• [Edmundo 03] $G' = [G, G]$ is nilpotent;
• [Baro-Jaligot-Ot.11] G' and G^k (lower central series) are definable for every k, and
• [Baro-Jaligot-Ot.11] G has an infinite abelian caracteristic subgroup.

Fact 2 [Frécon 00].
Let G be a definable definably connected nonnilpotent solvable group. Then, there is $N \trianglelefteq G$ definable and definably connected such that $(G/N)'$ is G/N-minimal (*i.e.* minimal among infinite definable normal subgroups of G/N) and $Z(G/N)$ is finite.
Proof of Fact 2. Applying Fact 1, we can follow the proof of Frécon in the finite Morley rank case. □

Proof of theorem 2: Solvable case

Minimal Configuration Lemma

Let G be a definable definably connected solvable group with G' G-minimal and $Z(G)$ finite. Then there is a definable $Q \leq G$ s. th.

- $G = G' \rtimes Q$ and $C_G(G') = G' \times Z(G) = F(G)$;
- Q is a abelian d. connected selfnormalizing Cartan of G;
- $C_G(x)$ is the only conjugate of Q containing x, for every $x \in G \setminus F(G)$;
- all complements of G' in G are G'-conjugate and largely generous Cartan subgroups of G; moreover, all Cartan subgroups of G are of this form.

Proof of theorem 2: Solvable case

Proof of Thm 2. By induction on $\dim G$.
WLOG, G is nonnilpotent.

- Apply Fact 2 to get a d. connected $N \trianglelefteq G$ s. th. $\overline{G}'$ is $\overline{G}$-minimal and $Z(\overline{G})$ is finite.
- Apply Minimal configuration Lemma to $\overline{G}$ to get a largely generous Cartan $\overline{H}$ of $\overline{G}$ with $\overline{G} = \overline{G}' \rtimes \overline{H}$.
- Apply I.H. to H to get Q largely generous Cartan of H.
- Now we have Q largely generous Carter of H and $\overline{H}$ largely generous in $\overline{G}$. From this we obtain (making use of the Weakly generousity lemma) that Q is largely generous Carter of G.
- The Minimal configuration lemma implies that Q is selfnormalizing and (3).
- Theorem 1 implies (4). □

Theorem 3: Semisimple case

Let G be a definable definably connected semisimple group. Then,

1. there exist Cartan subgroups of G;
2. there are only finite many conjugacy classes of Cartan subgroups of G;
3. if Q_1 and Q_2 are Cartan subgroups of G then, $Q_1^o = Q_2^o \Leftrightarrow Q_1 = Q_2$;
4. if Q is a Cartan subgroup of G then $Q' \leq Z(G)$ and $Q^o \leq Z(Q)$;
5. if Q is a Cartan subgroup of G then $\dim(Q^o) = \dim(G)$, and
6. the union of all Cartan subgroups of G is large in G.

Proof of theorem 3: Semisimple case

Transfer Lemma

Let G be a definably simple definable group. Then,

1. there exist Cartan subgroups of G;
2. there are only finite many conjugacy classes of Cartan subgroups of G;
3. Q_1 and Q_2 are Cartan subgroups of G then, $Q_1^o = Q_2^o \Leftrightarrow Q_1 = Q_2$;
4. Cartan subgroups are abelian;
5. if Q is a Cartan subgroup of G then $\dim((Q^o)^G) = \dim(G)$, and
6. the union of all Cartan subgroups of G is large in G.

Proof of theorem 3: Semisimple case

Fact 1.
Let G be a connected semisimple centreless Lie group. Then,

- Cartan subgroups of G are abelian;
- Q_1 and Q_2 are Cartan subgroups of G then,
  $Q_1^o = Q_2^o \Leftrightarrow Q_1 = Q_2$.

Fact 2 [Peterzil-Pillay-Starchenko 02].
Let G be a definably simple definable group. Then G is definably isomorphic to a semialgebraically simple group defined over the field of real algebraic numbers.

Proof of theorem 3: Semisimple case

Proof of the transfer lemma.

- Fact 2.
- By the facts on Lie groups, the properties (1)-(6) are satified, *mutatis mutandis*, by a connected semisimple centreless Lie group G.
- If G is definable over C and has finitely many conjugacy classes of Cartan subgroups then in each class there is a Cartan subgroup defined over C.
- Properties (1)-(6) are preserved under elementary extensions and elementary substructures.

□

Proof of theorem 3: Semisimple case

Fact 3 [Peterzil-Pillay-Starchenko 00].
Let G be a definably connected semisimple centreless group. Then G is definably isomorphic to the direct product of finitely many definably simple groups.

Proof of thm 3.

- Cartan subgroups of G are exactly the preimages in G of Cartan subgroups of $G/Z(G)$.
- Fact 3.
- Cartan subgroups of $G_1 \times \cdots \times G_s$ are exactly $Q_1 \times \cdots \times Q_s$, with Q_i Cartan subgroup of G_i ($1 \leq i \leq s$).
- Transfer lemma.

□

General case

Theorem 4: General case

Let G be a definable definably connected group. Then,

1. there exist Cartan subgroups of G;
2. there are only finite many conjugacy classes of Cartan subgroups of G;
3. if Q_1 and Q_2 are Cartan subgroups of G then, $Q_1^o = Q_2^o \Leftrightarrow Q_1 = Q_2$;
4. if Q is a Cartan subgroup of G then Q^o is weakly generous in G, and
5. the union of all Cartan subgroups of G has the same dimension than G.