A Categorical Approach to Internality

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Recent Developments in Model Theory
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Outline

Model Theoretic Internality

- Automorphism groups
- Internality

Categorical Reformulation

- Elimination of Imaginaries
- Stable Embeddings
- Internality

Automorphism groups

- $\mathcal{T}$ a theory
- M a model of $\mathcal{T}$
- $\mathrm{Aut}(M) = \{\tau : M \xrightarrow{\sim} M | \tau \text{ preserves all definable sets of } M\}$

Question

When is $\mathrm{Aut}(M)$ *definable?*

Example

Let $M = (\mathbb{C}, V)$, V a vector space of finite dimension over $\mathbb{C}$.
We have $r : \mathrm{Aut}(M) \to \mathrm{Aut}(\mathbb{C})$, so $\mathrm{Aut}(M)$ not definable.
But $\ker(r) = \mathrm{GL}(V)$ is definable.

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- $\mathrm{Aut}(M/M_0) = \{\tau \in \mathrm{Aut}(M) | \tau\restriction_{M_0} \text{ is the identity}\}$

Question

When is $\mathrm{G}(M) = \mathrm{Aut}(M/M_0)$ *definable?*

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Internality

All our theories eliminate imaginaries.

Definition ([Hrushovski, 2006])

An expansion $\mathcal{T}$ of $\mathcal{T}_0$ is an internal cover of $\mathcal{T}_0$ if $\mathcal{T}_0$ is stably embedded in $\mathcal{T}$, and any large model M of $\mathcal{T}$ contains a small set A with $\mathrm{dcl}(M_0 \cup A) = M$

Example

$M = (\mathbb{C}, V)$, $M_0 = \mathbb{C}$, A any basis of V over $\mathbb{C}$.

Theorem ([Hrushovski, 2006])

If $\mathcal{T}$ is an internal cover of $\mathcal{T}_0$, $M \mapsto \mathrm{Aut}(M/M_0)$ is definable: There is a (pro-) definable group $\mathbf{G}$ in $\mathcal{T}$, and a definable group action of $\mathbf{G}$ on every definable set of $\mathcal{T}$, identifying $\mathbf{G}(M)$ with $\mathrm{Aut}(M/M_0)$ for any model M.

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Categorical setting

Replace

- A theory $\mathcal{T}$ by the category $\mathcal{C} = \mathcal{D}_{\mathcal{T}}$ of definable sets and definable maps.
- Models of $\mathcal{T}$ by functors $M : \mathcal{C} \to \mathcal{S}ets$.
- Elementary maps by maps of functors
- Interpretations of $\mathcal{T}_0$ in $\mathcal{T}$ by functors $\mathcal{D}_{\mathcal{T}_0} \to \mathcal{D}_{\mathcal{T}}$.

[Makkai and Reyes, 1977]: Full characterisation of categories and functors arising this way. We need:

- $\mathcal{C}$ has all finite (inverse) limits and finite co-products.
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Elimination of Imaginaries

A theory $\mathcal{T}$ eliminates imaginaries if it satisfies the following equivalent conditions:

- Any definable equivalence relation has a quotient
- Any family of definable sets is equivalent to a canonical one
- For any definable sets $\mathbf{X}$ and $\mathbf{Y}$, there is an ind-definable set $\underline{\mathrm{Hom}}(\mathbf{X}, \mathbf{Y})$, with $\underline{\mathrm{Hom}}(\mathbf{X}, \mathbf{Y})(M)$ the set of M-definable functions from $\mathbf{X}$ to $\mathbf{Y}$.

In a general category $\mathcal{C}$, given objects $\mathbf{X}$ and $\mathbf{Y}$, define

$$\underline{\mathrm{Hom}}(\mathbf{X}, \mathbf{Y})(\mathbf{Z}) = \mathrm{Hom}(\mathbf{X} \times \mathbf{Z}, \mathbf{Y})$$

as a functor $\mathcal{C} \to \mathcal{S}ets$

Definition

The category $\mathcal{C}$ eliminates imaginaries if each $\underline{\mathrm{Hom}}(\mathbf{X}, \mathbf{Y})$ is ind-representable.

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A restatement of the Theorem

The existence of the definable Galois group can be stated in two steps.

- Any automorphism of M over M_0 is definable with parameters (from M).
- The assignment

$$\mathrm{G}(A) = \{A\text{-definable automorphisms of } M \text{ over } M_0\}$$

is definable.

For a functor $F : \mathcal{C} \to \mathcal{D}$, define

$$\mathbf{G}_F(\mathbf{T}) = \mathcal{A}\mathrm{ut}(\mathbf{T} \times F/\mathbf{T}) \tag{1}$$

(so $\mathbf{G}_F$ is a functor from $\mathcal{D}$ to $\mathcal{S}ets$)

Proposition

If $\mathcal{D}$ eliminates imaginaries, then $\mathbf{G}_F$ is a (pro-) group in $\mathrm{Ind}(\mathcal{D})$.

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Stable embeddings

Let $\mathcal{T}$ be a theory

- A definable set $\mathbf{X}$ in $\mathcal{T}$ is stably embedded if for any model M, any M-definable subset of $\mathbf{X}$ is definable over $\mathbf{X}(M)$
- A reduct $\mathcal{T}_0$ is stably embedded if all definable sets of $\mathcal{T}_0$ are stably embedded in $\mathcal{T}$
- Equivalently, for any two definable sets $\mathbf{X}$ and $\mathbf{Y}$ of $\mathcal{T}_0$, $\underline{\mathrm{Hom}}_{\mathcal{T}_0}(\mathbf{X},\mathbf{Y}) = \underline{\mathrm{Hom}}_{\mathcal{T}}(\mathbf{X},\mathbf{Y})$

If $F : \mathcal{C} \to \mathcal{D}$ is a functor between categories that eliminate imaginaries, evaluation induces a map

$$F(\underline{\mathrm{Hom}}(\mathbf{X},\mathbf{Y})) \to \underline{\mathrm{Hom}}(F(\mathbf{X}),F(\mathbf{Y})) \tag{2}$$

For any objects $\mathbf{X}$ and $\mathbf{Y}$ of $\mathcal{C}$

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Internality

Example

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An expansion $\mathcal{T}$ of $\mathcal{T}_0$ is an internal cover if

- It is a stable embedding
- There is a set A of parameters such that the restriction $(M, A) \mapsto M_0$ is an equivalence between models of $\mathcal{T}_A$ and of $\mathcal{T}_0$ (with inverse $M_0 \mapsto (\mathrm{dcl}(M_0 \cup A), A)$)
- Equivalently, the composition $\mathcal{T}_0 \to \mathcal{T} \to \mathcal{T}_A$ is a bi-interpretation

Definition

An internal cover of a category $\mathcal{C}$ is a category $\mathcal{D}$ and a pair of stable embeddings $I : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{C}$, together with an isomorphism of $F \circ I$ to the identity of $\mathcal{C}$

Internality

Example

If $\mathcal{T}$ is a theory, A a set of parameters, the expansion $\mathcal{T}_A$ is a stable embedding of $\mathcal{T}$

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Theorem

Let $I : \mathcal{C}_0 \to \mathcal{C}$, $F : \mathcal{C} \to \mathcal{C}_0$ be an internal cover. Then for any $H : \mathcal{C}_0 \to \mathcal{D}$, the map of groups

$$\mathrm{Aut}(H \circ F/I) \to \mathrm{Aut}(\Gamma_{\mathcal{D}} \circ H \circ F/I)$$

obtained by applying $\Gamma_{\mathcal{D}}$ is an isomorphism.

Note: H need not be closed in this theorem.

Corollary

Let $I : \mathcal{C}_0 \to \mathcal{C}$, $F : \mathcal{C} \to \mathcal{C}_0$ be an internal cover. There is a pro-group $\mathbf{G}$ in $\mathrm{Ind}(\mathcal{C})$, and an action of $\mathbf{G}$ on each object of $\mathcal{C}$, such that for any closed functor $H : \mathcal{C}_0 \to \mathcal{D}$ into an ind-closed category, $\Gamma_{\mathcal{D}}(H(F(\mathbf{G}))) = \mathrm{Aut}(\Gamma_{\mathcal{D}} \circ H \circ F/I)$.

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