Integer-valued functions and rational points on definable sets

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Recent Developments in Model Theory, Oléron
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Historical Background

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Theorem (Pólya 1920)

If $f : \mathbb{C} \to \mathbb{C}$ is entire and such that $f(\mathbb{N}) \subseteq \mathbb{Z}$, then, if

$$\limsup_{r\to\infty} \frac{m(f,r)}{2^z} < 1,$$

then f is a polynomial, where $m(f,r) := \sup\{f(z) : |z| \leq r\}$.

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This theorem has many descendants for functions in $\mathbb{C}$.
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The above does not apply in the real analytic setting;
consider, say, $f(x) = \sin(\pi x)$.

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Suppose that $f : \mathbb{R} \to \mathbb{R}$ *is definable in an o-minimal expansion of* $\overline{\mathbb{R}} := \langle \mathbb{R}, <, +, \cdot, 0, 1 \rangle$ *with the property that* $f(\mathbb{N}) \subseteq \mathbb{Z}$. *If there is a polynomial* $p \in \mathbb{R}[X]$ *such that ultimately* $f(x) < p(x)$, *then there is a polynomial* $q \in \mathbb{Q}[X]$ *such that ultimately* $f(x) = q(x)$.

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We shall prove a result in the direction of Pólya's for functions definable in $\mathbb{R}_{\exp} := \langle \overline{\mathbb{R}}, \exp\rangle$.
(It will, in fact, be applicable more generally.)

For a function f, let $M_f(r) := \sup\{|f(\bar{x})| : \bar{x} \in \overline{B_r(0)} \cap [0,\infty)^k\}$.

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Theorem (Jones-T.-Wilkie 2011)

Let $f\colon [0,\infty)^k \to \mathbb{R}$ be a function definable in $\mathbb{R}_{\exp}$, which is analytic and such that $f(\mathbb{N}^k) \subseteq \mathbb{Z}$. If, for all $\varepsilon > 0$, ultimately $M_f(t) < \exp(t^{\varepsilon})$, then f is a polynomial over $\mathbb{Q}$.

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For example,

$$f(x) = \exp_n(2\log_n(x)) \text{ and } g(x) = \exp_n(\frac{1}{2}\log_{n-1}(x))$$

are both definable in $\mathbb{R}_{\exp}$ and analytic, and both ultimately grow slower than $\exp(t^\varepsilon)$, for any $\varepsilon > 0$, but faster than all polynomials. (So $f(\mathbb{N}), g(\mathbb{N}) \not\subseteq \mathbb{Z}$.)

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If X contains "too many" rational points, then it must contain an infinite connected semialgebraic set.

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Consider $X^{\text{trans}} := X \setminus X^{\mathrm{alg}}$, the transcendental part of X, where X^{alg} is the union of all infinite, connected, semialgebraic subsets of X.
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But it is not a finitary/infinitary question - consider $\left|\text{graph}(2^x) \cap \mathbb{Q}^2\right|$. Not finite but 2^x is a transcendental function.

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Theorem (Pila-Wilkie 2006)

*Let $X \subseteq \mathbb{R}^n$ be definable in an o-minimal expansion of $\overline{\mathbb{R}}$.
For all $\varepsilon > 0$, there exists $c(X,\varepsilon) > 0$ such that, for all (sufficiently large) $T \in \mathbb{N}$,*

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Best possible statement for o-minimal expansions of $\overline{\mathbb{R}}$ in general (counterexample curve in $\mathbb{R}_{\text{an}}$).

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Wilkie's Conjecture (2006)

For all sets X definable in $\mathbb{R}_{\exp}$, there exist $c(X), \gamma(X) > 0$ such that

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Theorem (Jones-T. 2010)

For $f : I \longrightarrow \mathbb{R}$ existentially definable in $\mathbb{R}_{Pfaff}$, with $X := graph(f)$, there are $c(X), \gamma(X) > 0$ s.t. $|X^{trans} \cap \mathbb{Q}^n(T)| \leq c(\log T)^{\gamma}$, for $T \geq e$.

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In particular, this bound will hold for any function definable in any model complete reduct of $\mathbb{R}_{\text{Pfaff}}$ - in particular in $\mathbb{R}_{\exp}$.

Theorem (Jones-T. 2010; also Butler 2010)

Wilkie's Conjecture holds for any 1-dimensional set X.

Aside: Surfaces

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Combining methods of Pila for a certain surface with the 1-dimensional case and mild parameterization* for $\mathbb{R}_{\text{an}}$, we have:

Theorem (Jones-T. 2010)

If $X \subseteq \mathbb{R}^n$ is a surface definable in $\mathbb{R}_{resPfaff}$, the real field expanded by all restricted Pfaffian functions, then there exist $c(X), \gamma(X) > 0$ such that $|X^{trans} \cap \mathbb{Q}^n(T)| \leq c(\log T)^\gamma$, for all $T \geq e$.

Theorem (Jones-T. 2010)

Wilkie's Conjecture holds for any surface X which admits a mild parameterization.*

* Mild parameterization - a kind of covering by the images of finitely many functions with nice derivatives.

First, the one variable version of the theorem.

Proposition (Jones-T.-Wilkie 2011)

Let $f : (0,\infty) \to \mathbb{R}$ be a function definable in $\mathbb{R}_{\exp}$, which is analytic and such that $f(\mathbb{N}) \subseteq \mathbb{Z}$. If, for all $\varepsilon > 0$, ultimately $f(t) < \exp(t^{\varepsilon})$, then f is a polynomial over $\mathbb{Q}$.

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So suppose that f is transcendental.
Then we have $c(X),\gamma(X)>0$ s.t. $|X\cap\mathbb{Q}^n(T)|\leq c(\log T)^{\gamma}$, for $T\geq e$, where $X=\text{graph}(f)$. ctd...

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k-Variable Functions

Theorem (Jones-T.-Wilkie 2011)

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Proof of Theorem.

Let $f\colon [0,\infty)^k \to \mathbb{R}$ be analytic, definable in $\mathbb{R}_{\exp}$, have $f(\mathbb{N}^k) \subseteq \mathbb{Z}$ and $M_f(r) < \exp(r^{\varepsilon})$ ultimately, for all $\varepsilon > 0$.

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$$f(r,\bar{\theta}) = c_0(\bar{\theta}) + \ldots + c_{d(\bar{\theta})}(\bar{\theta}) r^{d(\bar{\theta})}, \text{ with } c_i(\bar{\theta}) \in \overline{\mathbb{Q}} \cap \mathbb{R}, d(\bar{\theta}) \in \mathbb{N}.$$

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Since the exponent map is definable, it is piecewise continuous (take a cell decomposition). It takes natural number values at directions with rational slope and is therefore constant on each open cell, with some bound $d(\bar{\theta}) \leq d \in \mathbb{N}$. ctd...

k-Variable Functions ctd

Theorem (Jones-T.-Wilkie 2011)

Let $f\colon [0,\infty)^k \to \mathbb{R}$ be a function definable in $\mathbb{R}_{\exp}$, which is analytic and such that $f(\mathbb{N}^k) \subseteq \mathbb{Z}$. If, for all $\varepsilon > 0$, ultimately $M_f(t) < \exp(t^\varepsilon)$, then f is a polynomial over $\mathbb{Q}$.

Proof of Theorem ctd.

We can apply the same process iteratively to $f(r,\bar{\theta}) - c_d(\bar{\theta})r^d$ etc. to show that f can be represented as $f(r,\bar{\theta}) = c_0(\bar{\theta}) + \ldots + c_d(\bar{\theta})r^d$, except possibly on a set of directions of lower dimension.

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Because f is analytic, the coefficients $c_i(\bar{\theta})$ are bounded, and hence f has polynomial growth in the radius.

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We can then show that it must be a polynomial over $\overline{\mathbb{Q}} \cap \mathbb{R}$ (and hence over $\mathbb{Q}$). □