Model theory of transseries

Lou van den Dries

University of Illinois

June 6, 2011, Oleron-meeting

Outline

I will describe a fascinating mathematical object, the field $\mathbb{T}$ of transseries. It is an ordered differential field extension of $\mathbb{R}$ and is a kind of universal domain for real differential algebra.

Conjecture: the elementary theory of $\mathbb{T}$ is model complete, and is the model companion of the theory of H-fields.

After discussing $\mathbb{T}$ we introduce H-fields, and then sketch some partial results towards this conjecture.

(Joint work with Aschenbrenner and van der Hoeven)

Reminder on Laurent series

The ordered differential field $\mathbb{R}((x^{-1}))$ of formal Laurent series in *descending* powers of x over $\mathbb{R}$ consists of all series of the form

$$f(x) = \underbrace{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x}_{\text{infinite part of } f} + \underbrace{a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots}_{\text{finite part of } f}$$

$x > \mathbb{R}$ for the ordering, $x' = 1$ for the derivation. *Defects*:

- x^{-1} has no antiderivative $\log x$ in $\mathbb{R}((x^{-1}))$.
- There is no natural exponentiation defined on all of $\mathbb{R}((x^{-1}))$; such an operation should satisfy $\exp x > x^n$ for all n.

Exponentiation does make sense for the *finite* elements of $\mathbb{R}((x^{-1}))$:

$$\begin{aligned} &\exp(a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots) \\ =&e^{a_0} \sum_{n=0}^{\infty} \frac{1}{n!}(a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots)^n \\ =&e^{a_0}(1 + b_1 x^{-1} + b_2 x^{-2} + \cdots) \end{aligned}$$

The field of transseries

To remove these defects we extend $\mathbb{R}((x^{-1}))$ to an ordered differential field $\mathbb{T}$ of *transseries*: series of *transmonomials* (or logarithmic-exponential monomials) arranged from left to right in decreasing order and multiplied by real coefficients, for example

$$e^{e^x} - 3e^{x^2} + 5x^{1/2} - \log x + 1 + x^{-1} + x^{-2} + x^{-3} + \cdots + e^{-x} + x^{-1}e^{-x} \ .$$

The reversed order type of the set of transmonomials that occur in a given transseries series can be any countable ordinal. (For the series displayed it is $\omega + 2$.) Such series occur for example in solving implicit equations of the form $P(x, y, e^x, e^y) = 0$ for y as $x \to +\infty$, where P is a polynomial in 4 variables over $\mathbb{R}$. The Stirling expansion for the Gamma function is also a transseries. Transseries also arise naturally as formal solutions to algebraic differential equations.

Transseries

Some typical computations in $\mathbb{T}$:

- **Taking a reciprocal**

$$\frac{1}{x - x^2 e^{-x}} = \frac{1}{x(1 - xe^{-x})} = x^{-1}(1 + xe^{-x} + x^2 e^{-2x} + \cdots)$$
$$= x^{-1} + e^{-x} + xe^{-2x} + \cdots$$

- **Formal Integration**

$$\int \frac{e^x}{x}\, dx = constant + \sum_{n=0}^{\infty} n! x^{-1-n} e^x \qquad (\text{ diverges}).$$

- **Formal Composition**

  Let $f(x) = x + \log x$ and $g(x) = x \log x$. Then

$$f(g(x)) = x \log x + \log(x \log x)$$
$$= x \log x + \log x + \log(\log x)$$

Transseries

- **Formal Composition continued**

$$
\begin{aligned}
g(f(x)) &= (x + \log x)\log(x + \log x) \\
&= x\log x + (\log x)^2 + (x + \log x)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{\log x}{x}\right)^n \\
&= x\log x + (\log x)^2 + \log x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\frac{(\log x)^{n+1}}{x^n}.
\end{aligned}
$$

- **Compositional Inversion**

  The transseries $g(x) = x\log x$ has a compositional inverse of the form

$$
\frac{x}{\log x}\left(1 + F\left(\frac{\log\log x}{\log x}, \frac{1}{\log x}\right)\right)
$$

  where $F(X, Y)$ is an ordinary convergent power series in the two variables X and Y over $\mathbb{R}$.

Properties of $\mathbb{T}$

Some key properties of $\mathbb{T}$: it is a real closed ordered field extension of $\mathbb{R}$, and is equipped with natural operations of *exponentiation* (exp) and (termwise) differentiation, $f \mapsto f'$, such that

$$\exp(\mathbb{T}) = \mathbb{T}^{>0}, \qquad \{f' : f \in \mathbb{T}\} = \mathbb{T}, \qquad \{f \in \mathbb{T} : \ f' = 0\} = \mathbb{R}.$$

As an exponential ordered field $\mathbb{T}$ is an elementary extension of the real exponential field. The iterated exponentials

$$x, \ \exp x, \ \exp(\exp(x)), \ldots$$

are cofinal in the ordering of $\mathbb{T}$.

Some Écalle quotes

It seems that the algebra $\mathbb{T}^{as}$ of accelero-summable transseries is truly the algebra-from-which-one-can-never-exit and that it marks an almost impassable horizon for "ordered analysis" . (This sector of analysis is in some sense “orthogonal” to harmonic analysis)

It seems (but I have not yet verified this in all generalityl) that $\mathbb{T}^{as}$ is closed under resolution of differential equations, or, more exactly, that if a differential equation has formal solutions in $\mathbb{T}$, then these solutions are automatically in $\mathbb{T}^{as}$

Cette notion de fonction analysable représente probablement l’extension ultime de la notion de fonction analytique (réelle) et elle parait inclusive et stable á un degre inouï

Conjectures about $\mathbb{T}$

From now on we consider $\mathbb{T}$ as an ordered differential field.

Conjecture 1: $\mathbb{T}$ is model complete.

Conjecture 2: If $X \subseteq \mathbb{T}^n$ is definable, then $X \cap \mathbb{R}^n$ is semialgebraic.

Conjecture 3: $\mathbb{T}$ is *asymptotically o-minimal*, that is, for each definable $X \subseteq \mathbb{T}$ either all sufficiently large $f \in \mathbb{T}$ are in X, or all sufficiently large $f \in \mathbb{T}$ are outside X.

Conjecture 4: $\mathbb{T}$ has NIP

Positive evidence

Asymptotic o-minimality holds for quantifier-free definable $X \subseteq \mathbb{T}$.

Best evidence for *model-completeness* of $\mathbb{T}$: the detailed analysis by van der Hoeven in "Transseries and Real Differential Algebra" (Springer Lecture Notes 1888) of the set of zeros in $\mathbb{T}$ of any given differential polynomial in one variable over $\mathbb{T}$. He proved:

Theorem

Given any differential polynomial $P(Y) \in \mathbb{T}\{Y\}$ and $f, h \in \mathbb{T}$ with $P(f) < 0 < P(h)$, there is $g \in \mathbb{T}$ with $f < g < h$ and $P(g) = 0$.

Here and later $K\{Y\} = K[Y, Y', Y'', \dots]$ is the ring of differential polynomials in the indeterminate Y over a differential field K.

Linear differential operators over $\mathbb{T}$

Another analogy with the real field is that linear differential operators over $\mathbb{T}$ behave much like one-variable polynomials over $\mathbb{R}$. A linear differential operator over $\mathbb{T}$ is an operator $A = a_0 + a_1\partial + \cdots + a_n\partial^n$ on $\mathbb{T}$ ($\partial =$ the derivation, all $a_i \in \mathbb{T}$); it defines the same function on $\mathbb{T}$ as the differential polynomial $a_0 Y + a_1 Y' + \cdots + a_n Y^{(n)}$. The linear differential operators over $\mathbb{T}$ form a noncommutative ring under composition.

Theorem

Each linear differential operator over $\mathbb{T}$ of order $n > 0$ is surjective as a map $\mathbb{T} \to \mathbb{T}$, and is a product (composition) of operators $a + b\partial$ of order 1 and operators $a + b\partial + c\partial^2$ of order 2.

The role of H-fields

Abraham Robinson taught us to think about model completeness in an algebraic way. Accordingly, we introduce a class of ordered differential fields, the so-called H-fields. These are defined so as to share certain basic (universal) properties with $\mathbb{T}$. The challenge is then to show that the "existentially closed" H-fields are exactly the H-fields that share certain deeper first-order properties with $\mathbb{T}$. If we can achieve this, then $\mathbb{T}$ will be model complete.

An H-field K is *existentially closed* if every differential polynomial over K with a zero in an H-field extension of K has a zero in K.

H-fields

Let K be an ordered differential field, and put

$$\begin{aligned} C &= \{a \in K : a' = 0\} && \text{(constant field of } K\text{)} \\ \mathcal{O} &= \{a \in K : |a| \leq c \text{ for some } c \in C^{>0}\} && \text{(convex hull of } C \text{ in } K\text{)} \\ \mathfrak{m}(\mathcal{O}) &= \{a \in K : |a| < c \text{ for all } c \in C^{>0}\} && \text{(maximal ideal of } \mathcal{O}\text{)} \end{aligned}$$

We call K an H-**field** if the following conditions are satisfied:

(H1) $\mathcal{O} = C + \mathfrak{m}(\mathcal{O})$,

(H2) $a > C \implies a' > 0$,

(H3) $a \in \mathfrak{m}(\mathcal{O}) \implies a' \in \mathfrak{m}(\mathcal{O})$.

Examples of H-fields: Hardy fields containing $\mathbb{R}$ such as $\mathbb{R}(x, e^x)$, the ordered differential field $\mathbb{R}((x^{-1}))$ of Laurent series, $\mathbb{T}$.

Liouville closed H-fields

Notation: $z^\dagger := z'/z$ for nonzero z in a differential field.

The real closure of an H-field is again an H-field. Call an H-field K *Liouville closed* if it is real closed and for all $a \in K$ there are $y, z \in K$ such that $y' = a$ and $z \neq 0$, $z^\dagger = a$. For example, $\mathbb{T}$ is Liouville closed. A *Liouville closure* of an H-field K is a minimal Liouville closed H-field extension of K.

Theorem

Each H-field has exactly one or exactly two Liouville closures.

Whether we have one or two Liouville closures is controlled by a key trichotomy in the class of H-fields. We discuss this in the next slide.

Trichotomy for H-fields

Any H-field K comes with a definable valuation v whose valuation ring is the convex hull $\mathcal{O}$ of C. Let Γ be the value group of v and $\Gamma^* := \Gamma \setminus \{0\}$. The derivation of K induces a function

$$\gamma = v(a) \mapsto \gamma' = v(a') \ : \ \Gamma^* \to \Gamma$$

and we put $\Gamma^\dagger := \{\gamma' - \gamma : \ \gamma \in \Gamma^*\}$. Then $\Gamma^\dagger < (\Gamma^{>0})'$, and exactly one of the following holds:

1. $\Gamma^\dagger < \gamma < (\Gamma^{>0})'$ for some (necessarily unique) γ;
2. $\Gamma^\dagger$ has a largest element;
3. $\sup \Gamma^\dagger$ does not exist; equivalently, $\Gamma = (\Gamma^*)'$

If $K = C$ we are in case 1, $\mathbb{R}((x^{-1}))$ falls under case 2, and Liouville closed H-fields under case 3. In case 1 there are two Liouville closures of K, and in case 2 there is only one.

Immediate Extensions of H-fields

An H-field falling under case 3 is said to admit asymptotic integration. For a long time we couldn't prove that every H-field has a case 1 extension. We only knew it for *maximally valued* H-fields with asymptotic integration. But two years ago we showed:

Theorem

Every real closed H-field with asymptotic integration has an immediate H-field extension that is maximally valued.

Complication: such an extension is not in general unique.

Corollary

Each H-field has a case 1 extension (and thus a case 2 extension).

Compositional Conjugation

Let K be an H-field, with derivation ∂. Typically, a differential polynomial $P \in K\{Y\}$ becomes more accessible to analysis when we make appropriate changes of variables. A key transformation of this kind is to rewrite P in terms of a derivation $\phi^{-1}\partial$ with $\phi \in K^\times$. The resulting differential field we call K^ϕ and P when rewritten in terms of the new derivation is P^ϕ, so $P^\phi \in K^\phi\{Y\}$. We restrict ϕ to be *admissible* in the sense that K^ϕ should still be an H-field; equivalently, $v\phi < (\Gamma^{>0})'$.

Theorem

Suppose K admits asymptotic integration and $P \in K\{Y\}$, $P \neq 0$. Then there is a differential polynomial $N(P) \in C\{Y\}$, $N(P) \neq 0$, such that for all admissible ϕ with sufficiently large $v\phi$ we have

$$P^\phi = aN(P) + R, \qquad a \in K^\times,\ R \in K^\phi\{Y\},\ v(R) > v(a).$$

We call $N(P)$ the Newton polyomial of P.

Differential-newtonian H-fields

An important fact about $\mathbb{T}$ is that if the Newton polynomial of $P \in \mathbb{T}\{Y\}$ has degree 1, then P has a zero in the valuation ring.

Define an H-field K to be *differential-newtonian* if it admits asymptotic integration and every nonzero $P \in K\{Y\}$ whose Newton polynomial has degree 1 has a zero in the valuation ring. Thus $\mathbb{T}$ is differential-newtonian.

If K is differential-newtonian, then every linear differential equation $a_0 y + a_1 y' + \cdots + a_n y^{(n)} = b$ over K has a solution in K.

There is also a related notion of *differential-henselian*: roughly, differential-newtonian is equivalent to certain coarsenings of compositional conjugates of K being differential-henselian.

Consequences for existentially closed H-fields

Using the results on the previous slides, many known results about $\mathbb{T}$ can now be shown to go through for existentially closed H-fields. For example, existentially closed H-fields are Liouville closed and differential-newtonian. In particular, every linear differential equation over an existentially closed H-field can be solved.

Let $A = a_0 + a_1\partial + \cdots + a_n\partial^n$ be a *linear differential operator* over the differential field K; here all $a_i \in K$, and ∂ stands for the derivation operator. These operators form a ring under composition, with $\partial a = a\partial + a'$ for $a \in K$.

Theorem

If K is existentially closed and $a_n \neq 0$, then $A : K \to K$ is surjective, and A is a product (composition) of operators $a + b\partial$ of order 1 and operators $a + b\partial + c\partial^2$ of order 2.

Simple Newton polynomials

The Newton polynomials of differential polynomials over $\mathbb{T}$ have the very special form

$$(c_0 + c_1 Y + \cdots + c_m Y^m) \cdot (Y')^n \qquad (c_0, \ldots, c_m \in \mathbb{R} = C).$$

This fails for some other H-fields with asymptotic integration, and we now understand better what makes it true. In $\mathbb{T}$ this has to do with the iterated logarithms ℓ_n with

$$\ell_0 = x, \quad \ell_{n+1} = \log \ell_n.$$

This sequence is coinitial in $\mathbb{T}^{>\mathbb{R}}$, and

$$-\ell_n^{\dagger\dagger} \;=\; \frac{1}{\ell_0} + \frac{1}{\ell_0 \ell_1} + \cdots + \frac{1}{\ell_0 \ell_1 \cdots \ell_n}.$$

Then $(-\ell_n^{\dagger\dagger})$ is a pc-sequence without a pseudolimit in $\mathbb{T}$. (It does have a pseudolimit $\sum_{n=0}^{\infty} \frac{1}{\ell_0 \ell_1 \cdots \ell_n}$ in an H-field extension of $\mathbb{T}$.)

Another important pseudocauchy sequence

Set $\varrho(b) := (b^\dagger)^2 - 2(b^\dagger)'$. Then

$$\varrho(\ell_n^\dagger) = \frac{1}{\ell_0^2} + \frac{1}{\ell_0^2\ell_1^2} + \cdots + \frac{1}{\ell_0^2\ell_1^2\cdots\ell_n^2}$$

also gives a pc-sequence without pseudolimit in $\mathbb{T}$. These facts can be converted into elementary properties of $\mathbb{T}$ that seem to be key to further model-theoretic analysis:

(A1) $\forall a \exists b \left[v(a - b^\dagger) \le vb < (\Gamma^{>0})' \right]$;

(A2) $\forall a \exists b \left[v(a - \varrho(b)) \le 2vb, \quad vb < (\Gamma^{>0})' \right]$.

A *trouble-free* H-field is one that is real closed, admits asymptotic integration, and satisfies (A1) and (A2).

Trouble-free H-fields

Theorem

For a real closed H-field with asymptotic integration, the following are equivalent

- *K is trouble-free;*
- *the Newton polynomial of any nonzero differential polynomial $P \in K\{Y\}$ has the form*

$$(c_0 + c_1 Y + \cdots + c_m Y^m) \cdot (Y')^n \qquad (c_0, \ldots, c_m \in C).$$

Corollary

Let K be a trouble-free H-field and $P \in K\{Y\}$, $P \neq 0$. Then there are $\alpha \in \Gamma$, $a \in K^{>C}$ and $m, n \in \mathbb{N}$ such that

$$C < y < a \implies v(P(y)) = \alpha + mvy + nvy'$$

for all y in all H-field extensions of K.

More on Trouble-freeness

Fact: Every existentially closed H-field is trouble-free.

Conjecture: if K is a trouble-free H-field, then it has a unique maximal immediate trouble-free H-field extension.

Optimistic Conjecture: An H-field K is existentially closed if and only if K is Liouville closed, differential-newtonian, and trouble-free.