Definably quotients of locally definable groups

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Setting

Let $\mathcal{M} = M_1 \sqcup M_2$ be an arbitrary structure, a disjoint union of two sorts (no maps between M_1 and M_2).

Problem

What are the definable (interpretable) groups in $\mathcal{M}$?
(say, in terms of M_1 and in M_2)

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Answer 1

$G = H_1 \times H_2$, where H_i is definable in M_i, $i = 1, 2$.

Answer 2

$G = (H_1 \times H_2)/F$, where F is a finite subgroup.

Answer 3

$$0 \longrightarrow H_1 \longrightarrow G \longrightarrow H_2 \longrightarrow 1$$

A central extension G of a definable group H_2 in M_2 by a definable group H_1 in M_1 (via, say a finite co-cycle $\sigma : H_2 \times H_2 \to H_1$).

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One more answer

$G = (\mathcal{H}_1 \times \mathcal{H}_2)/\Gamma$, where $\mathcal{H}_i$ is a locally definable group in $\mathcal{M}_i$ and Γ an infinite small, non-definable, subgroup. **But G is definable in $\mathcal{M}_1 \sqcup \mathcal{M}_2$!**

Definition

A locally definable group $\langle \mathcal{G}, \cdot \rangle$ (in an ω-saturated structure) is a countable directed union of definable sets $\mathcal{G} = \bigcup_n X_n \subseteq M^k$, such that (i) for every m, n, the restriction of multiplication to $X_m \times X_n$ is definable (and (ii) for every m, n there exists ℓ with $X_m \cdot X_n \subseteq X_\ell$, $X_n^{-1} \subseteq X_\ell$).

Example

G definable group, $e \in X \subseteq G$ a definable set, and $\mathcal{G} = \langle X \rangle$ the subgroup of G generated by X.

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Definable quotients

Definition

For $\Gamma \subseteq \mathcal{G}$, we say that $\mathcal{G}/\Gamma$ is *definable* (interpretable) if there exists a definable (interpretable) group G and a locally definable surjective homomorphism $\phi : \mathcal{G} \to G$.

Example

$\langle R, <, +, a\rangle$ a large ordered, divisible, abelian group.
Take $\mathcal{G} = \bigcup_n (-na, na)$, a locally definable subgroup.

$$\Gamma = \mathbb{Z}a \subseteq \mathcal{G}.$$

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$\mathcal{M}$ is an arbitrary κ-saturated structure.

Fact

For $\mathcal{G}$ a locally definable group, and $\Gamma \leqslant \mathcal{G}$ a small normal subgroup. The group $\mathcal{G}/\Gamma$ is definable (interpretable) in $\mathcal{M}$ **iff** there exists a definable $X \subseteq \mathcal{G}$ such that
(i) $\mathcal{G} = \Gamma \cdot X$
(ii) $X \cap \Gamma$ is finite.
(The group Γ is "a lattice in $\mathcal{G}$" and the set X is a "fundamental set").

Proof

IF: We assume $\mathcal{G} = \Gamma X$. The set XX^{-1} definable $\Rightarrow XX^{-1} \subseteq FX$ (for finite $F \subseteq \Gamma$) $\Rightarrow XX^{-1} \cap \Gamma \subseteq FX \cap \Gamma = F(X \cap \Gamma)$ is finite.
$\Rightarrow$ '$x_1\Gamma = x_2\Gamma$' is definable for $x_1, x_2 \in X$.
Similarly, the relation $x_1x_2\Gamma = x_3\Gamma$ is definable for $x_1, x_2, x_3 \in X$.
$\Rightarrow$ can define a group on X/Γ ($\cong \mathcal{G}/\Gamma$).

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Example in dimension 2

$\mathcal{M} = \langle R_1, <, + \rangle \sqcup \langle R_2^{>0}, <, \cdot \rangle$, the additive and multiplicative groups of two disjoint real closed fields.

$\mathcal{G} = \bigcup_n (-na, na) \times \bigcup_n (1/b^n, b^n) \subseteq R_1 \times R_2$

It is generated by the box $X = [-a, a] \times [1/b, b]$.

$\Gamma =$ the subgroup generated by the elements $(a, 1)$ and $(0, b)$.

The quotient $\mathcal{G}/\Gamma$ is definable, using the box X.

In this case $\mathcal{G}/\Gamma$ is isomorphic to a product of definable groups, but we may choose Γ differently with $\mathcal{G}/\Gamma$ definable but not a direct product (Strzebonski).

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Definition

$\mathcal{M}$ sufficiently saturated, $\mathcal{G}$ a locally definable group. A definable set $Y \subseteq \mathcal{G}$ is called *generic* if there is a small subset $A \subseteq \mathcal{G}$ such that $\mathcal{G} = A \cdot Y$.

Notation For $X \subseteq \mathcal{G}$ definable, write

$$X(n) = \overbrace{XX^{-1} \cdots XX^{-1}}^{n \text{ times}}.$$

Fact

Assume that $\langle \mathcal{G}, + \rangle = \bigcup_n X(n)$ is *abelian*. If a definable $Y \subseteq \mathcal{G}$ is generic then there exists a finitely generated subgroup $\Gamma \subseteq \mathcal{G}$ such that $\Gamma + Y = \mathcal{G}$.

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O-minimality

$\mathcal{M}$ o-minimal, sufficiently saturated.

Topology

If $\mathcal{G}$ is a locally definable group then it admits a manifold-like group topology, (Baro-otero: with countably many charts).

Setting

$\mathcal{G}$ abelian, generated by a definable subset $0 \in X \subseteq \mathcal{G}$.

$$\mathcal{G} = \bigcup_n X(n)$$

$$\dim \mathcal{G} := max_n \dim X(n)$$

Question

When does $\mathcal{G}$ has a definable quotient of the same dimension?

Theorem (Eleftheriou-P)

Assume that $\mathcal{G} = \bigcup_n X(n)$ is abelian, with $X \subseteq \mathcal{G}$ definable, definably compact, definably connected set.
If $\mathcal{G}$ contains a definable generic set then there exists a finitely generated subgroup $\Gamma \subseteq \mathcal{G}$ such that $\mathcal{G}/\Gamma$ is definable (definably compact) and $\dim(\mathcal{G}/\Gamma) = \dim \mathcal{G}$.

Negative example

Let R be a non-archimedean real closed field. $a >> 0$ in R. The group $\mathcal{G} = \langle \bigcup_n [-a^n, a^n], + \rangle$ does not have any definable generic subset!

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Proof of Theorem. $(1) \Rightarrow (2)$: **Easy direction**

We already saw: If $\mathcal{G}/\Gamma$ is definable then $\mathcal{G}$ contains a definable generic set

Proof. $(2) \Rightarrow (1)$

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We may assume that X is generic, so there is a finitely generated subgroup $\Gamma_0 \subseteq \mathcal{G}$ such that $X + \Gamma_0 = \mathcal{G}$.

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Digression-the type definable $\mathcal{G}^{00}$

Definition

A *type-definable subgroup of bounded index of* $\mathcal{G} = \bigcup X(n)$ is a subgroup H, contained in some $X(n)$ and given by a small intersection of definable sets, such that $[\mathcal{G} : H] < \kappa$.

Note: There are o-minimal, locally definable subgroups which have no type definable subgroups of bounded index (see previous "negative example")

Fact, H-P-P (a-la Shelah)

If $\mathcal{G}$ is locally definable in an NIP theory and if $\mathcal{G}$ contains **some** type-definable subgroup of bounded index then the intersection of ALL these subgroups, is type definable of bounded index. We call it $\mathcal{G}^{00}$.

Logic topology

A subset $F \subseteq \mathcal{G}/\mathcal{G}^{00}$ is closed if its pre-image in $\mathcal{G}$ is relatively type definable (namely, $\pi^{-1}(F) \cap X$ is type-definable for every definable $X \subseteq \mathcal{G}$).

Fact If $\mathcal{G}^{00}$ exists then $\mathcal{G}/\mathcal{G}^{00}$ is a locally compact.

Example

$$\mathcal{G} = \langle \bigcup_n [-n, n], + \rangle \subseteq \langle R, + \rangle$$

$$\mathcal{G}^{00} = \bigcap_n (-1/n, 1/n)$$

$$\mathcal{G}/\mathcal{G}^{00} \cong \langle \mathbb{R}, + \rangle$$

Logic topology

A subset $F \subseteq \mathcal{G}/\mathcal{G}^{00}$ is closed if its pre-image in $\mathcal{G}$ is relatively type definable (namely, $\pi^{-1}(F) \cap X$ is type-definable for every definable $X \subseteq \mathcal{G}$).

Fact If $\mathcal{G}^{00}$ exists then $\mathcal{G}/\mathcal{G}^{00}$ is a locally compact.

Example

$$\mathcal{G} = \langle \bigcup_n [-n, n], + \rangle \subseteq \langle R, + \rangle$$

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Main Theorem, Eleftheriou-P

Assume that $\mathcal{G} = \bigcup_n X(n)$ is an abelian group, with X definable, definably compact, definably connected. Assume that $X + \Gamma_0 = \mathcal{G}$ for a finitely generated Γ_0.

Then there exists $\Gamma \subseteq \Gamma_0$ such that $G := \mathcal{G}/\Gamma$ is a definable, definably compact group, and the following diagram commutes:

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Conjectures

$\mathcal{M}$ is o-minimal

Conjecture A

If $\mathcal{G}$ is abelian, generated by a definably connected set $X \ni 0$ then $\mathcal{G}$ always contains a definable generic subset (and therefore $\mathcal{G}$ always has definable quotient of same dimension).

Q: possible connection to approximate groups?

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Back to the original question

$\mathcal{M} = \langle M, <, +, \cdots \rangle$ a sufficiently saturated o-minimal expansion of an ordered group. By the Trichotomy Theorem, there are three possibilities:

1. $\mathcal{M}$ is a reduct of an ordered vector space (*the semilinear setting*).

2. $\mathcal{M}$ is an expansion of a real closed field (*triangulation theorem, Cohomology Theory etc*).

3. Some RCF's are definable, only on bounded intervals (*the semi-bounded setting*). No definable bijections between bounded and unbounded intervals.
Example: A reduct of a real closed field R: $\langle R, <, +, B \rangle$, with B a bounded semialgebraic subset of R^n.

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The idea

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Theorem (Eleftheriou -P)

Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, and G a definable, definably compact definably connected group. Then we have the following locally definable covering of G.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{G} & \longrightarrow & K & \longrightarrow & 1 \\ & & & & \downarrow & & & & \\ & & & & G & & & & \end{array}$$

with
(i) $\mathcal{H}$ a locally definable semilinear group.
(ii) K **a definable**, definably compact group in an o-minimal expansion of a real closed field.
(iii) $\mathcal{G}$ a locally definable, central extension, with $\dim(\mathcal{G}) = \dim(G)$.

Consequences

One can prove results about definable groups in o-minimal expansions of ordered groups using results about semilinear groups and about definable groups in o-minimal expansions of RCF's.

Example of such result

Compact Domination holds for definably compact groups in o-minimal expansions of ordered groups