The first-order theory of pseudoexponentiation

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Abstract

Zilber's axiomatization of pseudoexponentiation is in the logic $L_{\omega_1,\omega}(Q)$, which is necessary for a categoricity result. The first-order theory is more difficult to understand because of the presence of arithmetic. However, assuming the Conjecture of Intersections of Tori with subvarieties (CIT), we are able to separate out the effects of arithmetic and give an axiomatization of the complete first-order theory. This is joint work with Boris Zilber.

Pseudoexponential fields

Given by axioms capturing known and conjectural properties of

$$\mathbb{C}_{\exp} = \langle \mathbb{C}; +, \cdot, \exp \rangle$$

Definition

E-field $\langle F; +, \cdot, \exp \rangle$, field of characteristic zero, with homomorphism

$$\langle F; + \rangle \xrightarrow{\exp} \langle F^{\times}; \cdot \rangle$$

ELA-field Also F is algebraically closed, exp is surjective

- $\ker(F) = \{x \in F \mid \exp(x) = 1\}$, the kernel of the exponential map
- $Z(F) = \{r \in F \mid \forall x[x \in \ker \to rx \in \ker]\}$, its multiplicative stabilizer
- $\ker(\mathbb{C}) = 2\pi i\mathbb{Z}$ and $Z(\mathbb{C}) = \mathbb{Z}$

Definition

Standard kernel ker is an infinite cyclic group with transcendental generator τ.

Schanuel property and strong extensions

Schanuel Property (SP) The predimension function

$$\delta(\bar{x}) := \mathrm{td}(\bar{x}, \exp(\bar{x})) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x})$$

satisfies $\delta(\bar{x}) \geqslant 0$ for all tuples $\bar{x}$ from F.

On any E-field there is a pregeometry exponential-algebraic closure, ecl. When SP holds, the pregeometry arising from δ agrees with ecl.

A subset $A \subseteq F$ of an E-field is strong iff for every $\bar{x} \in F$

$$\delta(\bar{x}/A) := \mathrm{td}(\bar{x}, \exp(\bar{x})/A, \exp(A)) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x}/A) \geqslant 0$$

Strong Exponential-Algebraic Closedness (SEAC) F is existentially closed within the class of strong extensions which do not extend the kernel and which are exponentially algebraic.

Countable Closure Property (CCP) If A is countable then $\mathrm{ecl}(A)$ is countable.

Axioms

1. ELA-field – first-order
2. Standard kernel – $L_{\omega_1,\omega}$
3. Schanuel Property – first-order assuming 1 & 2
4. Strong exponential-algebraic closedness – first-order assuming 1, 2, & 3
5. Countable Closure Property – $L(Q)$-expressible
6. The exponential transcendence degree (ecl-dimension) is infinite – $L_{\omega_1,\omega}$

Definition

$\mathbf{ECF_{SK}}$ is given by axioms 1—4
$\mathbf{ECF_{SK,CCP}}$ is given by axioms 1—5
Both give natural abstract elementary classes.

Theorem (Zilber)

$\mathbf{ECF_{SK}} + 6$ *is a complete ($\aleph_0$-categorical) $L_{\omega_1,\omega}$-sentence.*
$\mathbf{ECF_{SK,CCP}} + Qx[x = x]$ *is a complete ($\aleph_1$-categorical) $L_{\omega_1,\omega}(Q)$-sentence.*

Aim: find the complete first-order theory of $\mathbf{ECF_{SK}}$.

First-order Axioms?

1 ELA-field

2 Standard kernel – $L_{\omega_1,\omega}$ – replace by:

2a ker is a cyclic Z-module

2b ker is transcendental over Z

2c′ Z is a model of the full first-order theory of $\langle \mathbb{Z}; +; \cdot \rangle$

3 Schanuel Property – problem

4 Strong exponential-algebraic closedness – problem

5 Countable Closure Property – not relevant to $L_{\omega_1,\omega}$ or first-order theory

6 Infinite exponential transcendence degree – not relevant to first-order theory

Failure of Schanuel Property

Suppose $r \in Z(F)$ is transcendental and $t \in \ker(F)$. Then $\{r^n t \mid n \in \mathbb{N}\}$ is $\mathbb{Q}$-linearly independent but of transcendence degree only 2, so

$$\delta(t, rt, r^2 t, \ldots, r^m t) = 2 - (m+1) = 1 - m < 0 \text{ for } m \geqslant 2$$

Strong Kernel

- Schanuel Property: for all $\bar{x}$, $\delta(\bar{x}/\emptyset) \geqslant 0$

The counterexample is inside the kernel. So instead we postulate the axiom

Strong kernel For all $\bar{x}$, $\delta(\bar{x}/\ker) \geqslant 0$

Strong kernel is equivalent to:

- If $(\bar{x}, \exp(\bar{x})) \in V \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$, subvariety defined over ker, and dim $V < n$, then $\exp(\bar{x})$ lies in a proper algebraic subgroup of $\mathbb{G}_m^n$.

Theorem

The strong kernel property is first-order axiomatizable if and only if the Conjecture on atypical Intersections of Tori with subvarieties (CIT) is true.

Proof idea.

Any $\exp(\bar{x})$ from a counterexample to Strong Kernel lies in some atypical intersection, but CIT says that such atypical intersections are definable. Conversely, take a counterexample to CIT and use a compactness argument to get a counterexample to Strong Kernel. □

The class $\mathbf{ECF_{StrK}}$

$\mathbf{ECF_{StrK}}$ is given by axioms

1 ELA-field

2′ first-order approximation of standard kernel

3′ Strong kernel

4 Strong exponential-algebraic closedness

- SEAC is first-order axiomatizable assuming 1, 2′, and 3′
- So $\mathbf{ECF_{StrK}}$ is a first-order theory assuming CIT
- Assuming 1, 2′, 3′, and CIT, SEAC is equivalent to exponential-algebraic closedness: for every $n \in \mathbb{N}^+$ and every rotund subvariety V of $\mathbb{G}_a^n \times \mathbb{G}_m^n$ there is $\bar{x}$ in F such that $(\bar{x}, e^{\bar{x}}) \in V$.
- SEAC also requires $(\bar{x}, e^{\bar{x}})$ to be generic in V over any given finite set of parameters.

Rotundity is a first-order definable property of V saying that every $\mathbb{Q}$-linear projection of V has suitably large dimension.

Main Results

- Within $\mathbf{ECF_{StrK}}$ there is a notion of saturation over the kernel: saturation within the subclass of $\mathbf{ECF_{StrK}}$ where the kernel does not extend.

Unconditional Lemma

Suppose $F \in \mathbf{ECF_{StrK}}$. Then for each cardinal $\lambda \geqslant |F|$, there is $F \subseteq M$ with $|M| = \lambda$, $\ker(M) = \ker(F)$, and M saturated over the kernel.

Unconditional Theorem – Superstability over the kernel

Suppose $F, M \in \mathbf{ECF_{StrK}}$, with $\aleph_0$-saturated kernel, both saturated over the kernel and of the same cardinality. Suppose that we have an isomorphism $\theta_Z : Z(F) \cong Z(M)$.
Then there is an isomorphism $\theta : F \cong M$ extending θ_Z.

Theorem (assuming CIT)

$\mathbf{ECF_{StrK}}$ is a complete first-order theory.

Corollaries, assuming CIT

Z is stably embedded in $\mathbf{ECF_{StrK}}$.

$\mathbf{ECF_{StrK}}$ has quantifier elimination in the expansion with symbols for every definable subset of $\mathbb{Z}$ and every existential formula – near model completeness over the kernel.

If $\theta_Z : Z_1 \hookrightarrow Z_2 \models \mathrm{Th}(\mathbb{Z})$ there are $F_i \models \mathbf{ECF_{StrK}}$ with $Z(F_i) = Z_i$, and $\theta : F_1 \hookrightarrow F_2$ extending θ_Z. If θ_Z is elementary then θ can be chosen to be elementary.

We know (unconditionally) that $\mathbf{ECF_{StrK}}$ is not model complete even after adding symbols for every definable subset of $\mathbb{Z}$. However, with CIT we get that the non-model completeness of $\mathbb{Z}$ also gives a proof of non-model completeness of $\mathbf{ECF_{StrK}}$.