Simplicity and pseudofiniteness

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Stable theories

- A canonical ideal of "small" formulas, relative to a given type. (forking).
- Finite rank (or superstable): a dimension theory $dim : Def \to \mathbb{N}/Ord$; a dimension theory on types. a/Ab small $\iff$ $\phi(a,b)$ holds, $\phi(x,b)$ small relative to $tp(a/A)$ $\iff$ $\dim(a/Ab) < \dim(a/A)$.
- 2-amalgamation.
- *unique, up to the profinite Galois action.*
- Shelah analyzes isomorphism types; Zilber, geometry; etc., all based on unique 2-amalgamation. E.g. Zilber's *stabilizer*, a definable subgroup associated with a definable subset X of a group G. If $aX \cap X$ is not small, then the symmetric difference is small; such elements a form a subgroup, the stabilizer.

Simple theories

- An ideal of "small" formulas, relative to a given type. (forking).
- Finite rank (or supersimple): a dimension theory $dim : Def \to \mathbb{N}/Ord$; a dimension theory on types. a/Ab small $\iff$ $\phi(a,b)$ holds, $\phi(x,b)$ small relative to $tp(a/A)$ $\iff$ $\dim(a/Ab) < \dim(a/A)$.
- 3-amalgamation.
- Kim-Pillay spaces; compact Lascar types.
- Geometric simplicity theory constructed on this basis. Example: stabilizer.

The compact Lascar group

- algebraic closure: the union of finite (=bounded) A-definable sets, *including imaginaries*
- Galois group G_{pf} = image of $Aut(\mathbb{U})$ in $Sym(acl(A))$.
- Galois correspondence: closed subgroups of G_{pf} - substructure of $acl(A)$.
- bdd = continuous alg. closure: the union of bounded A-definable sets of hyperimaginaries, i.e. D/E where $E = \cap_n Y_n$, Y_n definable.
- Compact Lascar group G_c = image of $Aut(\mathbb{U})$ in $Sym(bdd(A))$.
- D/E has a natural topology, where U is open iff the pullback in D is a union of definable sets. Induces a *compact group* structure on G_c.

The compact quotient of a definable group

- A duality: automorphism groups, definable groups.
- G^{00} = minimal subgroup of bounded index. G/G^{00} a similar compact topological group structure. For *Ind*-definable $\tilde{G}$, a locally compact topology.

3- amalgamation

- 1-skeleton data: types $p_i(x_i)$.
- 2-skeleton data: types $p_{ij}(x_i, x_j)$, *free*.
- compatibility: $p_i \subset p_{ij}$.
- Solution: p_{123} containing each p_{ij}; *free*.

The Galois obstruction

- For a *finite* set, 3-amalgamation fails.
- In fact for a *bounded set*; hence for any set with a bounded invariant quotient.
- In particular, the compact Lascar group measures an obstruction to 3-amalgamation.
- Kim-Pillay show that for simple theories, this is the *only* obstruction.

Pseudo-finite theories (T, δ, μ_D)

- $T = \lim_u T_i$.
- An ideal of "small" (=measure 0) formulas, relative to a given formula or (almost every) type. $\mu_D(P) = \lim_u |P|/|D| \in \mathbb{R}^{\infty}_{\geq 0}$
- A dimension theory on (nonempty) definable sets. $\delta(D) = \lim_u \log |D| + Conv(\mathbb{R}) \in \mathbb{R}^* / Conv(\mathbb{R})$.
- Let $D' \subset D$. Then $\delta(D') < \delta(D)$ iff $\mu_D(D') = 0$.
- Canonical real-valued quotients of V near $\delta(D)$: $\overline{\delta}_D(X) = lim_u |X|/|D| \in \mathbb{R}^{\infty}_{\geq 0}$.
- 3-amalgamation.

Coarse pseudo-finite dimension: properties of $\overline{\delta} = \overline{\delta}_D$

$\overline{\delta}(Y) \in \mathbb{R}^{\geq 0}_{\infty}$ for nonempty definable Y. If $\Gamma = \cap Y_n$, $Y_1 \supset Y_2 \supset \ldots$, let $\delta(\Gamma) = \inf \delta(Y_n)$.

- $\overline{\delta}(\{y\}) = 0$.
- $\overline{\delta}(Y \cup Y') = \max(\overline{\delta}(Y), \overline{\delta}(Y'))$
- $\overline{\delta}(Y \times Y') = \overline{\delta}(Y) + \overline{\delta}(Y')$
- More generally, if f is a definable function on Y,

  $$\overline{\delta}(Y) = \inf\{\alpha + \beta : \alpha \in \mathbb{R}_{\infty}, \beta = \dim\{z : \overline{\delta}(f^{-1}(z)) \geq \alpha\}$$

  This holds for $Y \to Y/E$ even for an $\bigwedge$-definable equivalence relation T.
- Write $Y_a = f^{-1}(a)$. Then for any $\alpha < \beta \in \mathbb{R}$, $\{a : \overline{\delta}(Y_a) \leq \alpha\} \subset D \subset \{a : \overline{\delta}(Y_a) < \beta\}$ for some definable a.

3-amalgamation for definable measures (v1)

- 1-skeleton data: types $p_i(x_i)$.
- 2-skeleton data: types $p_{ij}(x_i, x_j)$, *free*.
- compatibility: $p_i \subset p_{ij}$.
- For *almost all* $(p_i), (p_{ij})$, there exists p_{123} containing each p_{ij}; and p_{123} avoids any definable measure-zero set.

Proofs of 3-amalgamation

- Till recently, only one proof was known to model theorists. It was for simple theories, and based on 2-uniqueness in stability, and stability of the relation: "$\phi(x, a) \& \psi(x, b)$ is small".
- This proof most naturally yields 3-*replacement*: if amalgamation data has a solution p_{ijk}, weakly random k/ij, then the same is true if p_{12} is replaced by p'_{12}. Requires *weak randomness* only, i.e. the ideal of *definable sets* of measure 0. But angle-amalgamation must be obtained separately.
- Generalize to $n \geq 3$ using higher forking.

Proofs of 3-amalgamation

- Proof by Towsner of n-amalgamation for measures, over a model. Roots go back to Roth's proof of Szemeredi's theorem, for $n = 3$; "energy increment method." At the same time, Towsner's proof is (independently) isomorphic to the proof for stable theories enriched by automorphisms (pseudo-finite fields, small PAC fields, ACFA). (picture).
- Related statement: triangle removal. 3-amalgamation problem as an intersection of 3 partial types. *assuming* 3-amalgamation is possible, there exists
  - Does not involve Galois obstruction.
  - Can be viewed as an instance of dimension theorem.

Stabilizer lemma

- Current proof of stabilizer theorem uses 3-replacement. Smoother proof using 3-amalgamation?
- In additive combinatorics, 3-amalgamation and stabilizer lemma corresponds to known but nontrivial results (triangle removal, Szemeredi lemma, Balog-Szemeredi,. . ., Sanders.)
- A potential two-way connection:
  - Locally compact groups, $\widetilde{G}/G^{00}$. (cf. Gromov; cf. Furstenberg, in amenable setting.)
  - Relative triangle removal.
  - Modularity, trichotomy.

Definable and $\bigwedge$-definable groups in pseudo-finite theories

G a definable group, $S = \cap_n Y_n$ be an ∞-definable subgroup.

- S has strict dimension $\alpha \in V$ if $\delta(Y_n) = \alpha$ for large n.
- Expected: if S has strict dimension, then (up to finite index), $N \leq S \leq H$, $N \trianglelefteq H$, H/N nilpotent. (Known when $G_0 \prec G$ is finitely generated, and $G_0 \leq S$.)
- (?) Definably simple groups are ultraproducts of finite simple groups. In particular, either of finite rank or one exponent away from it. (cf. John Wilson.)

Definable and $\bigwedge$-definable groups

G a *linear* definable group in a pseudo-finite theory.

- if an ∞-definable S has strict dimension, then up to finite index, $N \leq S \leq H$, $N \triangle H$, H/N nilpotent. In fact this holds if $0 < \overline{\delta}(S) < \infty$ [Breuillard-Green-Tao, Pyber-Szabo, 2010]
- (Jordan 1878). If G has no unipotent elements, then G is finite.
- (Larsen-Pink) If G (or the Zariski closure of G) is simple, then G is definable over pseudo-finite field with automorphism; G is of Lie type.

Definable and $\bigwedge$-definable linear groups

- If $0 < \overline{\delta}(S) < \infty$, then $N \leq S \leq H$, $N \triangle H$, H/N nilpotent. [Breuillard-Green-Tao, Pyber-Szabo, 2010]
- (Jordan 1878). If G is linear and has no unipotent elements, then G is finite.
- (Larsen-Pink) If G is linear, say simple, then G is definable over pseudo-finite field with automorphism; G is of Lie type.

CFSG trichotomy

A large finite simple group is:
Alt_n,
or an object of algebraic geometry
e.g. $SL_4(F_q)$,
or of high-dimensional linear algebra
$SL_n(F_2)$
or a combination of the two parameters $SL_n(F_q)$

- Follows from "classification of sporadics".
- Challenge: a pseudo-finite proof; effective in above sense.
- Implies much about *primitive finite structures* A,via $Aut(A)$; but *ineffectively*, in terms of complexity of formula defining equivalence relation. Hence, no direct consequences for primitive pseudo-finite structures.
- Properties of *all* structures equivalent to classification of primitive ones? (Gorenstein.)

Quasi-finite theories

Theorem (Cherlin-H., slightly updated)

Let T be pseudo-finite. Assume:

- *T is $\aleph_0$-categorical.*
- *Modularity: if A, B, C are algebraically closed in T^{eq}, $A \cap B = C$, and $a \in A$, then $\delta(a/B) = \delta(a/C)$.*
- *Every definable subset of an Abelian group is a Boolean combination of cosets, and an A_0-definable set.*
- *T does not interpret: (i) the generic graph, (ii) (V, I) where I is a generic subset of the dual of a vector space V.*

Then T is coordinatized by classical geometries over finite fields.

Quasi-finite theories

- Zilber's theory of envelopes extends to this setting (assuming interpretable orthogonal spaces are oriented.) The simple groups involved in the automorphism groups of the envelopes are, up to finitely many exceptions, Alt_n and groups of Lie type over a bounded finite field.
- Converse known to be true, using CSFG.
- Problem: Direct proof of modularity. Trichotomy assuming $\aleph_0$-categoricity and $\delta(Def) \cong \mathbb{Z}$? (Dugald Macpherson, Charlie Steinhorn, measurable structures.)
- Conditions preserved under interpretations; notably the reduct to relations of standard finite length; implies effective classification of this class of structures.
- Problem: reformulate as: $\aleph_0$-categoricity, modularity imply coordinatization by one of a number of concrete geometries.

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Seven (more) open issues

- weak randomness vs. randomness: omit a set of types of measure 0, but *open* or G_δ ?
- 2-skeleton randomness vs. 1+2-skeleton randomness. (Connections to "triangle removal".)
- *Relative* triangle removal and the dimension theorem?
- Base set: model? invariant type? Lascar types? (Caveat: for many combinatorial applications, Skolemization can be assumed, trivializing all Galois groups. Still.)
- Group configuration: (measure-theoretic formulation:) given an operation with associativity holding 1% of the time, even up to a correspondence, show that it is isomorphic to 1% of a group. Kim, dePiro Milar in simple theories with 4-amalgamation.

Higher amalgamation

- n-amalgamation: types p_w ($|w| < n$) in variables x_w; $x_{w'} \subset x_w$ for $w' \subset w$; seek $p_{\cup w}$.
- Will not be discussed in this talk. See papers by subsets of: Evans, Goodrick, Kim, de Piro- Millar, Kolesnikov, Tsuboi.
- 3-amalgamation / (hyper)imaginares = n-amalgamation / ?
- For *stable* theories, glimpses of understanding. Here the only constraint for 3-amalgamation is the algebraic imaginaries; for 4, groupoid imaginaries: given a definable isomorphism type, add an (imaginary) ideal representative. With a shift of 1, this goes through for simple theories
- For simple theories, not so simple? (GKT)
- A resonance with Lurie's higher toposes; from "equivalence relations in $\mathcal{C}$ are effective (Giraud)" to "every groupoid object of $\mathcal{X}$ is effective".

The compact Lascar group in pseudo-finite theories.

- $G^{00} = \cap_n X_n$, X_n definable, where $X_{n+1}X_{n+1} \subset X_n$ while finitely many translates of X_{n+1} cover X_n; approximate subgroups.
- An ∞-definable equivalence relation: $E = \cap_n R_n$, $R_{n+1} \circ R_{n+1} \subset R_n$; approximate equivalence relations..
- For each a complete type P, P/E is a homogeneous space for the compact Lascar group.
- In pseudo-finite setting, G^{00} is expected to be contained in a definable H, with H/G^{00} nilpotent.
- Do approximate equivalence relations arise from ∞-definable equivalence relations?
- Which compact groups can be realized as compact Lascar groups in pseudo-finite theories?

internality; beyond boundedness

- $\mathcal{C}$ a family of (hyper)definable sets. $\mathcal{D} = \{A$-(hyper)-definable sets internal to $\mathcal{C}$. $Aut(D/\mathcal{C})$ a pro-definable group.
- When $\mathcal{C} = \{finite/compact\}$, this is the profinite / compact Lascar group.
- Given a a definable group G, consider the smallest normal subgroup $N_{\mathcal{C}}$ with G/N internal to $\mathcal{C}$.
- $N_{Finite} = G^0$, $N_{compact} = G^{00}$.
- Generalize theory to other classes $\mathcal{C}$.
- Let $\mathcal{C} = \{X : \delta(X) << \delta(G)\ \}$.
  $1 \in X = X^{-1} \subset G, XX = XF, F \in \mathcal{C}$; is X bounded modulo $N_{\mathcal{C}}$?

Embedded dimension theories

- Recall trichotomy conjecture for reducts of ACF.
- A reduct of ACF is strongly minimal, and induces a dimension theory on varieties. (Cf. H.-Wagner, quasi-dimensions.)
- So does a pseudo-finite expansion of the theory of fields by a pseudo-finite X: $V \subset \mathbb{A}^m \mapsto \overline{\delta}_X(V \cap X^m)$.
- More general problem: dimension theories on varieties. Trichotomy.
- Larsen-Pink inequality: for $X = \Gamma$ an ∞-definable subgroup of G, $\overline{\delta}(V) \leq \frac{\dim(V)}{\dim(G)} \overline{\delta}(\Gamma)$.
- This approach (with Larsen-Pink inequality) is the model-theoretic input to Breuillard-Green-Tao's theorem on linear approximate groups.
- Conjectures of Bukh-Tsimerman (almost a reduct problem), Erdös -Szemeredi.

Embedded dimension theories, cont'd

Let $X \subset F_p$, $\overline{\delta} = \overline{\delta}_X$, $\overline{\delta}(\mathbb{F}_p) \geq 2$.

- Bukh-Tsimerman conjecture: Let $R \subset \mathbb{A}^3$ be a surface with finite projections to each $\mathbb{A}^1$. If $\overline{\delta}(R) = 2$, then $(\mathbb{A}^1, R, X)$ is isogenous to $(E, +, Y)$ with E a one-dimensional algebraic group, and Y to an approximate subgroup.
- Erdös -Szemeredi conjecture: $\overline{\delta}(X + XX) = 2\overline{\delta}(X)$.

The Larsen-Pink inequality

Proposition

Assume Γ is a Zariski dense subgroup of G, G a simple algebraic group. Let V be a subvariety of G. $\delta(V \cap \Gamma) \leq \frac{\dim(V)}{\dim(G)}\delta(\Gamma)$.

Proof.

(sketch for $\dim(V) = 1$, $\dim(G) = 2$.) We may assume V is irreducible. Define $f : (V \cap \Gamma)^2 \to G$, $f(y_1, h_2) = y_1 y_2^{-1}$. For $c \notin Stab(V)$, $f^{-1}(c)$ is finite. Hence $\delta(\Gamma) \geq \delta(f(\Gamma \cap Y)^2) \geq 2\delta(Y)$. □

Corollary

Let $a \in \Gamma$, $H = C_G(a)$. Then $\delta(\Gamma \cap H) = \frac{\dim(Y)}{\dim(G)}\delta(\Gamma)$.

This is obtained using the map $ad_a(x) = x^{-1}ax$; we have $\delta(a^G) = \delta(G) - \delta(H)$.

Proof of BGT

- $X[0] = X, X[n+1] = XX[n]$ $(n \in \mathbb{N}.)$
- to show: for any $0 < \epsilon < \epsilon'$, for some m, for all $X \subseteq G$ generating G, $|X[m]| \geq |X|^{1+\epsilon}$ (unless $|X|^{1+\epsilon'} > |G|$.)
- Suppose not. Then by compactness, can find $X_n (n \in \mathbb{Z})$ with $X_n X_n \subset X_{n+1}$ and $1 \leq \delta(X_n) \leq 1 + \epsilon < 1 + \epsilon' \leq \delta(G)$ for all n; and X_n contained in no definable subgroup of G.
- Let $\Gamma = \cap_n X_n$. This is a Zariski dense subgroup of G, $0 < \delta(\Gamma) < \infty$. Renormalize so that $\delta(\Gamma) = \dim(G)$.
- Let R be the set of regular semisimple elements of G. Note: $\dim(G \smallsetminus R) < \dim(G)$, so $\delta(\Gamma \smallsetminus R) < \delta(\Gamma)$.
- Let $\Upsilon = \{C_G(a) : a \in R \cap \Gamma\}$. Clearly, Υ is Γ-conjugation invariant. We will show Υ is definable, i.e. $\{b : C_G(b) \in \Upsilon\}$ is definable, using a dimension gap:

Proof of BGT

- Let $T = C_G(b)$, $b \in R$.
- $T = C_G(a), a \in R \cap \Gamma$, then $\delta(\Gamma \cap T) \geq \dim(T)$ by Larsen-Pink.
- If $\delta(T \cap X) > \dim(T) - 1$, then as $\delta((T \cap X)/(T \cap \Gamma)) \leq \delta(X/\Gamma) \leq \delta(X) - \delta(\Gamma) = 0$ we have:
- $\delta(T \cap \Gamma) > \dim(T) - 1 \geq \dim(T \setminus R)$ so $T \cap \Gamma \cap R \neq \emptyset$.
- Thus $T \in \Upsilon$ iff $\delta(T \cap X) > \dim(T) - 1$ iff $\delta(T \cap X) \geq \dim(T)$; so Υ is definable.
- Hence the normalizer $N(\Upsilon)$ is a definable group, and it contains Γ. By assumption, $N(\Upsilon) = G$.
- Fix $T \in \Upsilon$. $G/N(T)$ embeds into Υ; so $\delta(G/N(T)) \leq \delta(\Upsilon) = \delta(\Gamma) - \delta(N(T) \cap \Gamma)$. It follows that $\delta(G) = \delta(\Gamma) = \delta(X)$; contradicting the assumption on X.

Quasi-finite structures

L a finite language (e.g. graphs).

Theorem (Zilber, CHL; envelopes)

Let M be an infinite structure with $|M^k|/Aut(M) = f(k) < \infty$. Assume $\dim(Def(M)) \to \mathbb{N}$ (or Ord) is defined, with Morley dimension properties. Then it is possible to interpret in M a finite number of infinite dimensional projective geometries over finite fields, $V_1, \ldots, V_l$. M is a approximated by a family of finite structures $M(d) = M(d_, \ldots, d_l)$, with $\dim V_i(M(d)) = d_i$. For any sentence θ true in M and any $K \in \mathbb{N}$, for large enough d, $M(d) \models \theta$ and $M(d) \in C(L, f|K)$.

Example

$(\mathbb{Z}/4\mathbb{Z})^\infty$.

Quasi-finite structures

$C(L, f)$ = class of finite L-structures A such that $|A^k/Aut(A)| \leq f(k)$. $\bar{C}(L, f)$=first order closure.

Example

Classical geometries over finite fields: vector spaces with unitary/orthogonal/symplectic forms;

Definition

$a \downarrow_C b$ if $\delta(ab/C) = d(a/C) + \delta(b/C)$.

(Agrees with nonforking definition.)