Unexpected imaginaries in valued fields with analytic structure

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notation

In a valued field K with valuation $v : K \to \Gamma$, write

- $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ = valuation ring,
- $\mathfrak{m} = \{x \in K : v(x) > 0\}$ = maximal ideal,
- $k = \mathcal{O}/\mathfrak{m}$ = residue field,
- $\mathrm{RV} = K^*/(1 + \mathfrak{m})$.

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- $\mathrm{RV} = K^*/(1+\mathfrak{m})$.

Recall that $1 \to k^* \to \mathrm{RV} \to \Gamma \to 0$ is a short exact sequence:

$$\begin{aligned} b(1+\mathfrak{m}) = b'(1+\mathfrak{m}) &\iff b'/b \in 1+\mathfrak{m} \\ &\iff v(b'/b - 1) > 0 \\ &\iff v(b-b') > v(b'). \end{aligned}$$

quantifier elimination

Fix $\mathcal{L}_v$ a language for valued fields and with respect to which the appropriate theory has quantifier elimination. We can include the valuation for example with a div relation:

$$\text{div}(x, y) \iff v(x) \leq v(y).$$

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- pCF $\mathcal{L}_v = (+, \cdot, 0, 1, \{P_n\}, \mathrm{div})$, where $P_n(x) \iff \exists y(y^n = x)$; theory has QE by A. Macintyre (1976)

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- RCVF $\mathcal{L}_v = (+, \cdot, 0, 1, <, \mathrm{div})$; theory has QE by Cherlin-Dickmann (1983)

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Define an equivalence relation on linearly independent n-tuples from K^n by $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ if and only if $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ generate the same $\mathcal{O}$-submodule of K^n. The sort S_n is the sort of the equivalence classes of this equivalence relation; $\mathcal{S} = \bigcup_n S_n$.

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Notice that

- Γ is identified with S_1; γ is identified with the equivalence class s of elements a of K with $v(a) = \gamma$.

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For each $s \in S_n$ there is $A_s \subset K^n$ which is the $\mathcal{O}$-module coded by s. Define an equivalence relation on A_s by $a \sim b$ if and only if $a - b \in \mathfrak{m}A_s$. We write $\mathrm{red}(s)$ for the set of equivalence classes of this equivalence relation, and the sort T_n is the union of all $\mathrm{red}(s)$ for $s \in S_n$; $\mathcal{T} = \bigcup_n T_n$. Thus an element t of T_n codes the subset of the field $a + \mathfrak{m}A_s$, where $a \in A_s$.

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- RV is identified with the subset of T_1 given by $\{t \in T_1 : t \text{ codes } a + \mathfrak{m}A_s \text{ for } a \in A_s \setminus \mathfrak{m}A_s\}$.

elimination of imaginaries

The following theories have elimination of imaginaries in $(\mathcal{L}_v, \mathcal{G})$, for the appropriate $\mathcal{L}_v$ as described above.

- ACVF Haskell–Hrushovski–Macpherson (2006)
- RCVF Mellor (2006)
- pCF (In this case, the T_n sorts are not needed.) Hrushovski–Martin (arxiv)

analytic structure: p-adics

Let

$$\mathcal{A}_n = \{f \in \mathbb{Z}_p[[X_1, \ldots, X_n]] : \text{ coefficients have valuation converging to } \infty\}.$$

If $f \in \mathcal{A}_n$, then f defines a function $\mathbb{Z}_p^n \to \mathbb{Z}_p$.

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If $f \in \mathcal{A}_n$, then f defines a function $\mathbb{Z}_p^n \to \mathbb{Z}_p$.
Write $\mathcal{L}_{v,\text{an}}$ for the language $\mathcal{L}_v$ with function symbols for every function defined by a power series in $\mathcal{A} = \bigcup n\mathcal{A}_n$.
Interpret each function symbol by the function on $\mathbb{Z}_p$ defined by the power series.
Write pCF$^{\text{an}}$ for the theory of $\mathbb{Q}_p$ in $\mathcal{L}_{v,\text{an}}$.

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Denef–van den Dries (1988)

pCF$^{\text{an}}$ is model complete, and has quantifier elimination in $\mathcal{L}_{v,\text{an}}$ with a symbol added for a partial division function to the valuation ring.

analytic structure: algebraically closed valued field

Let K_0 be a complete rank one valued field. Let

$$\mathcal{A}_n = \{f \in K_0[[X_1, \ldots, X_n]] : \text{ coefficients have valuation converging to } \infty\}.$$

If $f \in \mathcal{A}_n$, then f defines a function $\mathfrak{m}^n(K_0) \to K_0$. For functions on $\mathcal{O}^n(K_0)$, we require tighter restrictions on the rate of convergence of the power series. Use two sorts of variables, ranging over the valuation ring and the maximal ideal; still write $\mathcal{A}$ for this more restricted collection of power series.

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Write $\mathcal{L}_{v,\text{an}}$ for the language $\mathcal{L}_v$ with function symbols for every function defined by a power series in $\mathcal{A}$, and ACVF$^{\text{an}}$ for the theory of K_0 in $\mathcal{L}_{v,\text{an}}$.

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Lipshitz (1993)

ACVF^{an} is model complete, and has quantifier elimination in $\mathcal{L}_{v,\text{an}}$ with symbols added for partial division functions to the valuation ring and maximal ideal.

analytic structure: real closed valued field

Write $\mathcal{L}_{\mathrm{an}}$ for the language of real closed fields with function symbols for every function which is analytic in a neighborhood of $[-1, 1]^n$ (interpreted in $\mathbb{R}$ by the *restricted* analytic function which is 0 outside of $[-1, 1]^n$) and in which the theory of $\mathbb{R}^{\mathrm{an}}$ is universally axiomatised. Let K be a non-standard model of the theory of $\mathbb{R}$ in $\mathcal{L}_{\mathrm{an}}$, and let $\mathcal{L}_{\mathrm{an},v}$ be a language with a predicate for the set of finite elements (a valuation ring) of K. Then the theory of K in this language is an example of a T*-convex* theory; call it $\mathrm{RCVF}^{\mathrm{an}}$.

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van den Dries–Lewenberg (1995)

RCVF^{an} has quantifier elimination in $\mathcal{L}_{\text{an},v}$.

analytic quantifier elimination

Furthermore, each of the above analytic theories is minimal in the appropriate sense; that is, in all models of the theory, definable sets in one variable are quantifier-free definable with the ‘minimal’ predicates:

- RCVF^{an}: the valuation and the ordering (weakly o-minimal) van den Dries–Lewenberg (1995)
- ACVF^{an}: just the valuation (C-minimal) Lipshitz–Z. Robinson (1998)
- pCF^{an}: the P_n predicates and the field structure (P-minimal) van den Dries–Haskell–Macpherson (1999)

analytic quantifier elimination

Generalization of all of the above settings provided by Cluckers–Lipshitz (2010)

Form a ring of quotients of power series $\mathcal{A}$ by:

- Begin with a ring of power series in arbitrarily many variables (possibly split into two sorts)
- Close under composition, restricted division
- Close under Weierstrass preparation

Add function symbols to the language for the functions on $\mathcal{O}^m \times \mathfrak{m}^n$ defined by the function symbols in $\mathcal{A}$.

Cluckers–Lipshitz develop the theory of analytic functions on a quasi-affinoid domain relative to $\mathcal{A}$.

One important result is that an analytic function on a K-domain can be written as a unit times a rational function.

This result used to prove quantifier elimination (much as Weierstrass preparation is used in the classic Denef–van den Dries style argument).

analytic elimination of imaginaries?

Question

Does the analytic theory have elimination of imaginaries in $(\mathcal{L}_{v,\mathrm{an}}, \mathcal{G})$?

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Answer: (Haskell-Hrushovski-Macpherson)

No.

Example of an $(\mathcal{L}_v, \mathcal{A})$-definable imaginary which is not coded in $\mathcal{G}$ (provided $\mathcal{A}$ contains the power series for restricted exponential and logarithm).

We'll go through the example carefully for $\mathrm{pCF}^{\mathrm{an}}$, and indicate briefly the differences for $\mathrm{ACVF}^{\mathrm{an}}$.

where to look for such an imaginary?

- at least two variables
- be analytically, and not algebraically, defined
- have some group structure

exponentiation

Note that the power series

$$G(X) = \sum_{n=0}^{\infty} \frac{p^n}{n!} X^n$$

has coefficients with valuation converging to ∞ and hence is in $\mathcal{A}_1$.
Also

$$\exp(x) = G(p^{-1}x) \text{ for any } x \in \mathfrak{m}$$

so the function $\exp : \mathfrak{m} \to 1 + \mathfrak{m}$ is $\mathcal{L}_{v,\mathrm{an}}$-definable.
Furthermore, the graph of exp

$$\{(x, \exp(x)) : x \in \mathfrak{m}\}$$

is a subgroup of $(\mathfrak{m}, +) \times (1 + \mathfrak{m}, \cdot)$.
To construct a definable set which is not coded, just need to take this set and make it more generic.

moving between sorts and subsets of the field

Fix $\mathcal{M}$ an ω-saturated model of $\mathrm{pCF}^{\mathrm{an}}$, $\mathcal{U}$ a monster model.
For any set of parameters C, write $\mathcal{O}(C) = \mathrm{dcl}(C) \cap \mathcal{O}$, and so on for all other sorts.

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Fix $\gamma \in \Gamma(\mathcal{M})$ (so $\gamma = v(c)$ for some $c \in K(\mathcal{M})$) with γ greater than all integers. Write

$$W = \mathcal{O}(\mathcal{M})/\gamma\mathcal{O}(\mathcal{M})$$

and for $w \in W$, write

$$A_w = a + \gamma\mathcal{O} = \{x \in K : v(x - a) \geq \gamma\}.$$

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Notice that, if $w \in W$ with $w \notin \mathrm{acl}(C)$, then there are no elements of A_w which are algebraic over Cw (by P-minimality).

moving between sorts and subsets of the field

For any $r \in \mathrm{RV}$, write

$$B_r = b(1+\mathfrak{m}) = \{x \in K : v(x-b) > v(b)\}.$$

Let q be the $\mathrm{Aut}(\mathcal{U})$-invariant partial type determined by the formulas $x > \delta$ for all $\delta \in \Gamma(\mathcal{U})$.

Now assume that $r \in \mathrm{RV}$ is such that $v(r) \models q$.

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If $r \notin \mathrm{acl}(C)$ then there are no elements of B_r which are algebraic over Cr (by P-minimality).

the example

Define an affine homomorphism from A_w to B_r by:

$$\begin{aligned} h_{ab} : A_w &\rightarrow B_r \\ h_{ab}(x) &= b\exp(pc^{-1}(x-a)) \end{aligned}$$

The graph of h_{ab} is

$$\{(a+y, b\exp(pc^{-1}y) : y \in \gamma\mathcal{O}\} = \{(a,b) * (y, \exp(pc^{-1}y)) : y \in \gamma\mathcal{O}\}$$

and thus is a coset of a subgroup of $(\gamma\mathcal{O}, +) \times (1+\mathfrak{m}, \cdot)$.

Then the graph of h_{ab} is not coded in $\mathcal{G}$.

justification of the example

Suppose for contradiction that there is a finite tuple in $\mathcal{G}$ which is a code for the graph of $h = h_{ab}$. This tuple must be (w, r, e_1, e_2) where e_1 is a finite tuple from K, e_2 is a finite tuple from the other sorts.

justification of the example: h coded in K over w, r

Let $C \supseteq \mathrm{acl}(\mathcal{M}wr)$ be such that there is another affine homomorphism $g : A_w \to B_r$ with the same homogeneous component as h and defined over C. Then $g(x) = b' \exp(pc^{-1}(x - a'))$ where $a' \in A_w(C)$, $b' \in B_r(C)$.

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Consider the function $\log(g/h) : A_w \to \mathfrak{m}$, where $\log : 1 + \mathfrak{m} \to \mathfrak{m}$ is the inverse of exp.

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$$\begin{aligned} \log(g/h)(x) &= \log\left(\frac{b' \exp(pc^{-1}(x-a'))}{b \exp(pc^{-1}(x-a))}\right) \\ &= \log(b/b') + (a - a') \\ &= d \in \mathfrak{m}. \end{aligned}$$

Thus $g(x) = \exp(d)h(x)$, so h is coded over C by $\exp(d) \in K$.

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The map $e_2 \to \exp(d)$ is C-definable from the sort of e_2 to K.
But any definable map from a non-field sort to the field has finite image and hence cannot be a code. Thus e_2 is in $\mathrm{acl}(\mathcal{M}wre_1)$.

justification of the example: one of two cases

Furthermore, $\dim(e_1/\mathcal{M}) = \dim(d/C) = 1$, so by Skolem functions, we may assume that $e_1 = e$ is a single field element.
(Dimension here is in the sense of model theoretic algebraic closure, which has the exchange property in $\mathrm{pCF}^{\mathrm{an}}$.)

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Case 1: $w \notin \operatorname{acl}(\mathcal{M}e)$

Case 2: $w \in \operatorname{acl}(\mathcal{M}e)$

justification of the example: case 1 $w \notin \mathrm{acl}(\mathcal{M}e)$

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For otherwise, we would have $w \in \operatorname{acl}(\mathcal{M}ev(r))$, hence (using Skolem functions) $w \in \operatorname{dcl}(\mathcal{M}ev(r))$. But then there would be a definable function from the ordered set Γ to the anti-chain W, which is not possible by P-minimality.

Choose $b' \in B_r$ so that $w \notin \operatorname{acl}(\mathcal{M}eb'r)$.

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Choose $b' \in B_r$ so that $w \notin \operatorname{acl}(\mathcal{M}eb'r)$.
Then no element of A_w is algebraic over $\mathcal{M}eb'rw$.

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Choose $b' \in B_r$ so that $w \notin \mathrm{acl}(\mathcal{M}eb'r)$.
Then no element of A_w is algebraic over $\mathcal{M}eb'rw$.
But $h^{-1} \in \mathrm{acl}(\mathcal{M}erw)$, so $h^{-1}(b')$ is an algebraic element of A_w over $\mathcal{M}eb'rw$. Contradiction.

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For if not, then $v(r)$ is finite with respect to $\Gamma(\mathcal{M}e)$; that is, $v(r) = v(d)$ for some $d \in K(\mathrm{acl}(\mathcal{M}e))$. Using algebraic exchange and some care, get $w \in \mathrm{acl}(\mathcal{M})$, contrary to hypothesis.

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Hence $v(r) \notin \mathrm{acl}(\mathcal{M}ew)$, so $r \notin \mathrm{acl}(\mathcal{M}ew)$.

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Hence $v(r) \notin \operatorname{acl}(\mathcal{M}ew)$, so $r \notin \operatorname{acl}(\mathcal{M}ew)$.

Choose $a' \in A_w$ so that $r \notin \operatorname{acl}(\mathcal{M}ea'w)$.

justification of the example: case 2 $w \in \mathrm{acl}(\mathcal{M}e)$

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Choose $a' \in A_w$ so that $r \notin \mathrm{acl}(\mathcal{M}ea'w)$.
Then no element of B_r is algebraic over $\mathcal{M}ea'wr$.
But $h \in \mathrm{acl}(\mathcal{M}ewr)$, so $h(a')$ is an algebraic element of B_r over $\mathcal{M}ea'wr$.
Contradiction.

justification of the example: conclusion

Since both possible cases lead to a contradiction, h cannot be coded.

modifications for ACVF

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Case 1 res$(w) \notin \mathrm{acl}(\mathcal{M}e)$, where $\mathrm{res}(w) = \mathrm{res}(x)$ for any $x \in A_w$.

Then also $w \notin \mathrm{acl}(\mathcal{M}er)$. (Careful argument using C-minimality.)

Case 2 res$(w) \in \mathrm{acl}(\mathcal{M}e)$

Then $r \notin \mathrm{acl}(\mathcal{M}ew)$. (Careful argument using C-minimality.)

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Case 2 res$(w) \in \mathrm{acl}(\mathcal{M}e)$

Then $r \notin \mathrm{acl}(\mathcal{M}ew)$. (Careful argument using C-minimality.)

Each case leads to a contradiction, as the function picks out an algebraic element of A_w (case 1) or B_r (case 2).

Questions

- If $\mathcal{A}$ does not include the exponential and logarithm functions, does the theory have EI to the sorts $\mathcal{G}$?
- What new sorts are required to eliminate imaginaries in the analytic setting?