Some applications of diophantine geometry and model theory to group theory.

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Université Paris-Sud, Orsay, France

Oléron, June 6th, 2011

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3. A uniform Tits alternative.
4. Diameter of finite simple groups.
5. Effective versions of Hrushovski's theorems on approximate groups.

Effective Burnside-Schur

Theorem (Restricted Burnside Problem; Kostrikin, Zelmanov)

Given natural integers r, n, there are only finitely many r-generated finite groups all of whose elements have order dividing n.

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For a symmetric set of generators $S = \{1, s_1^{\pm 1}, ..., s_r^{\pm 1}\}$ of a group $G = \langle S \rangle$, we denote by $S^k := S \cdot ... \cdot S$ the "ball of radius k" in the Cayley graph of G generated by S.

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Conjecture (Effective restricted Burnside; Olshanskii)

Given r, n, does there exists $k(r, n) \in \mathbb{N}$ such that there are only finitely many r-generated finite groups $G = \langle S \rangle$ such that all elements in S^k have order dividing n ?

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Remark (Olshanskii). The conjecture would solve a celebrated open problem of Gromov, i.e. show the existence of a non-residually finite Gromov hyperbolic group.

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Corollary (1st effective version)

Olshanski's conjecture holds for subgroups of GL_d (d fixed). That is: given r, n, d, there exists $k(r, n, d) \in \mathbb{N}$ such that there are only finitely many r-generated finite groups $G = \langle S \rangle$ admitting an embedding in GL_d (over some field) such that all elements in S^k have order dividing n.

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Then consider the ultraproduct $\prod_{\mathcal{U}} \langle S_k \rangle \subset \prod_{\mathcal{U}} \mathrm{GL}_d(K_k) = \mathrm{GL}_d(K)$, where

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In turn, this forces $\langle S_k\rangle$ to be bounded independently of k, contrary to the standing assumption. QED.

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Open pb: find optimal upper bounds for $K(n)$ (proof shows $K(n) \leqslant e^{n^{C(d)}}$).

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Recall the definition of the logarithmic Weil height on $\overline{\mathbb{Q}}^\times$. Let $K \leqslant \overline{\mathbb{Q}}^\times$ be a number field.

$$\text{For } x \in K, \text{ set } h(x) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in V_K} n_v \log^+ |x|_v,$$

where, as usual, V_K =set of places of K, $n_v = [K_v : \mathbb{Q}_v]$, and $\log^+ = \max\{\log, 0\}$.

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Definition

We call **normalized height** the quantity

$$\widehat{h}(S) := \lim_{n \to +\infty} \frac{1}{n} h(S^n),$$

where $S^n = S \cdot \ldots \cdot S$ is the n-th fold product set.

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(ii) (Bogomolov-type Height Gap Theorem) $\exists \varepsilon = \varepsilon(d) > 0$ *such that, unless* $\langle S \rangle$ *is virtually solvable, we have*

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(iii) (Comparison with heights of eigenvalues) if $\langle S \rangle$ *is Zariski-dense in* $\mathbf{G}$*, then for some* $C = C(d), c = c(d) > 0$,

$$\frac{1}{C}\widehat{h(S)} \leqslant \max\{h(\lambda); \lambda \text{ eigenvalue of some } g \in S^c\} \leqslant C\widehat{h}(S).$$

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5) The Bogomolov conjecture for tori (Zhang's theorem).

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Remark: The proof uses the Tits "ping-pong method" (extending earlier work of Eskin-Mozes-Oh and Breuillard-Gelander) and relies crucially on the Bogomolov-type result for $\widehat{h}(S)$ presented above.

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(ii) is a countable union of algebraic conditions, each equivalent to

$S \in \mathcal{W}_n :=$ “every two words in S^N have a relation of length $\leqslant n$”

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$$S \in \mathcal{V} := \text{“}\langle S \rangle \text{ has a solvable sugroup of finite index”}$$

(ii) is a countable union of algebraic conditions, each equivalent to

$S \in \mathcal{W}_n :=$ “every two words in S^N have a relation of length $\leqslant n$”

Then Uniform Tits reads:

Theorem (Uniform Tits Alternative, B. '08)

There are $N = N(d), M = M(d) \in \mathbb{N}$ such that if S is a finite symmetric set in $\mathrm{GL}_d(\mathbb{C})$, then
(i) either $\langle S \rangle$ has a solvable subgroup of index at most M,
(ii) or S^N contains two elements a, b, such that $\langle a, b \rangle$ is a non-abelian free group.

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So $\mathcal{V} = \mathcal{W}_n$ in char p for $p > p(n) \simeq exp(n^A)$.

In particular we have proved:

Corollary (uniform exponential growth)

There are $e, c, N, M > 0$ depending on d only such that if $S \subset \mathrm{GL}_d(\mathbb{F}_p)$ and $\langle S \rangle$ has no solvable subgroup of index at most M, then
(i) $\exists a, b \in S^N$ with no relation up to length $(\log p)^e$.
(ii) $|S^n| > \exp(cn)$ for every $n \leqslant (\log p)^e$.

Applications: diameter of finite simple groups

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Conjecture (Uniform logarithmic diameter for FSG's of Lie type, folklore)

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In fact, it is not obvious to find even one infinite family $\{G_n\}$ of finite groups for which $diam(G_n) \ll \log |G_n|$ (a question Lubotzky's)

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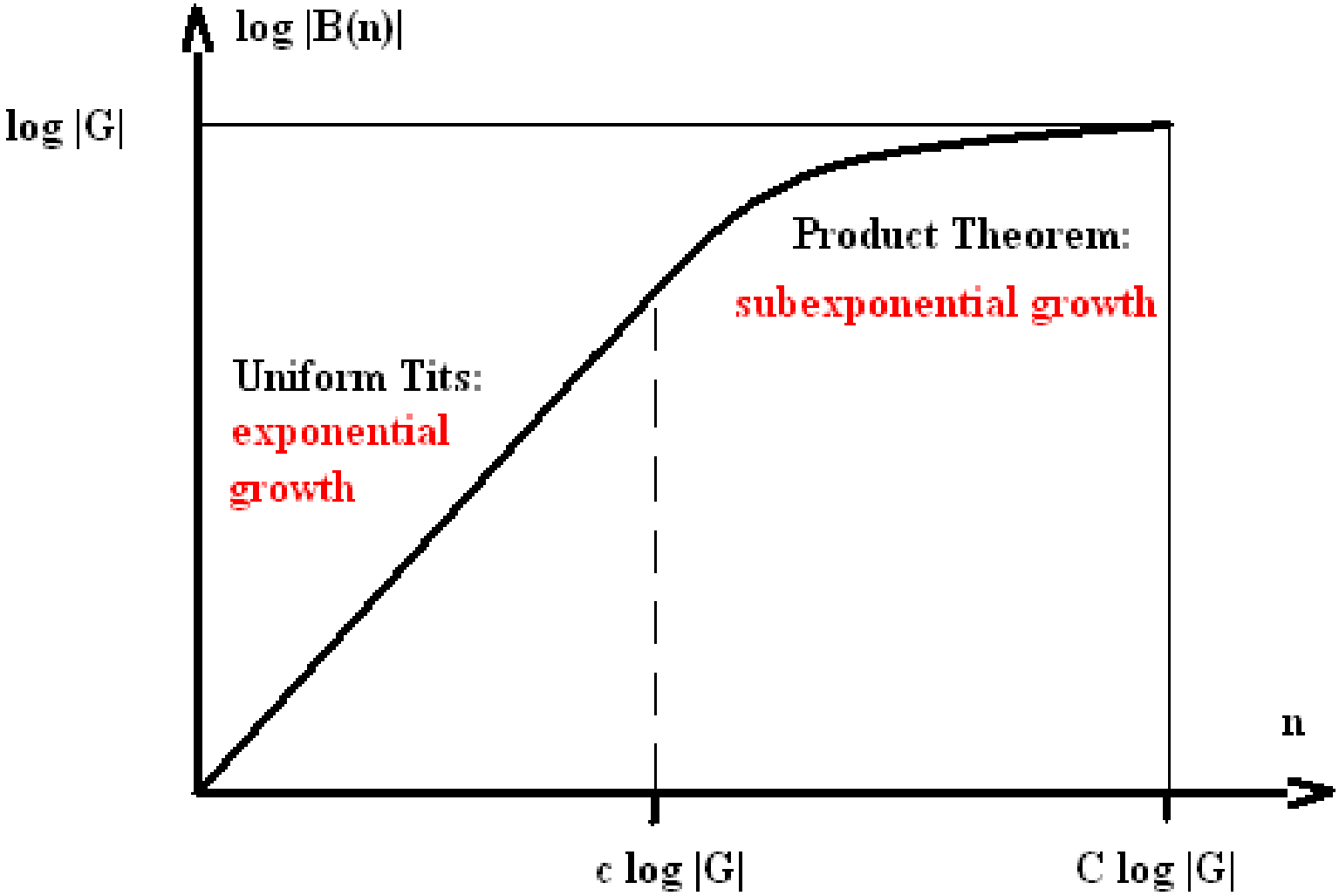
then we reach $B(n) = G$.



Growth of the ball of radius n, B(n), in a finite simple group of Lie type G.

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Open problem: can one take $p(n) \leqslant \exp(Cn)$? (currently we know $p(n) \leqslant exp(n^C)$).

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Let $r \in \mathbb{N}$. There exists a constant $\gamma = \gamma(r) > 0$ such that

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b) that $diam(G) \leqslant (\log |G|)^C$ for some $C > 0$ depending only on $rank(G)$... but not enough for $\leqslant C \cdot \log |G|$!

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The aforementioned subsequent works of Pyber-Szabo and Breuillard-Green-Tao aim at giving a better bound on how close to a genuine group is a given approximate group, resulting in the above stated theorem.

References: Heights on character varieties, uniform Tits

E. Breuillard, *A Height Gap Theorem for finite subsets of* $\mathrm{GL}_d(\overline{\mathbb{Q}})$ *and non amenable subgroups*, to appear Annals of Math (2011).

E. Breuillard, *A strong Tits alternative*, preprint, arXiv:0804.1395.

E. Breuillard, *Heights on* SL_2 *and free subgroups*, Zimmer Volume, Chicago Univ. Press (2011).

E. Breuillard and A. Gamburd, *Strong uniform expansion in* $\mathrm{SL}(2,p)$, Geom. Anal. Func. Anal. Vol. 20-5 (2010), 1201-1209.

References: Approximate groups and applications to finite groups

E. Breuillard, B. J. Green and T. C. Tao, *Approximate subgroups of linear groups*, to appear Geom. Anal. Func. Anal. (2011).

H. A. Helfgott, *Growth and generation in* $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, Ann. of Math. (2) **167** (2008), no. 2, 601–623.

E. Hrushovski, *Stable group theory and approximate subgroups*, preprint (2009), arXiv:0909.2190.

L. Pyber and E. Szabó, *Growth in finite simple groups of Lie type*, preprint (2010), arXiv:1001.4556.

Thank you!