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Weak One-Basedness

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based on work of Alexander Berenstein and Evgueni Vassiliev

with thanks Alexander Berenstein, Evgueni Vassiliev, Rizos Sklinos

and

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[Some attributions of definitions and results were given in the talk but are not included in these slides.]

Let M be sufficiently saturated.
Let $T = Th(M)$.
Let $\downarrow$ be an independence relation on M.

<u>Axioms</u>: Given $\bar{a} \in M^n$, $B \subseteq C \subseteq D \subseteq M$,

Invariance: If $\sigma \in Aut(M)$ and $\bar{a} \downarrow_B C$ then $\sigma(\bar{a}) \downarrow_{\sigma(B)} \sigma(C)$.

Symmetry: $\bar{a} \downarrow_B C$ iff, $\forall$ finite $\bar{c} \subseteq C$, $\bar{c} \downarrow_B B\bar{a}$.

Transitivity: $\bar{a} \downarrow_B D$ iff $\bar{a} \downarrow_B C$ and $\bar{a} \downarrow_C D$.

Extension: There is some $C' \equiv_B C$ such that $\bar{a} \downarrow_B C'$.

Local character: There is some $E \subseteq acl(B)$ such that $|E| \leqslant 2^{2^{|T|}}$ and $\bar{a} \downarrow_E BE$.

Anti-reflexivity: If $\bar{a} \downarrow_B B\bar{a}$ then $\bar{a} \in acl(B)$.

Definition:

- $tp(\bar{a}/B)$ is one-based if, $\forall \bar{a}' \equiv_B \bar{a}$, $\bar{a} \downarrow_B \bar{a}' \Rightarrow \bar{a} \downarrow_{\bar{a}'} B$.
- $tp(\bar{a}/B)$ is weakly one-based if $\exists C \supseteq B$ such that $\bar{a} \downarrow_B C$ and $tp(\bar{a}/C)$ is one-based.
- $tp(\bar{a}/B)$ is very weakly one-based if $\exists \bar{a}' \equiv_B \bar{a}$ such that $\bar{a} \downarrow_B \bar{a}'$ and $\bar{a} \downarrow_{\bar{a}'} B$.
- $tp(\bar{a}/B)$ is weakly locally modular if $\exists C$ such that $C \downarrow_B \bar{a}$, $C \downarrow_{\bar{a}} B$ and $\bar{a} \downarrow_{acl(C\bar{a}) \cap acl(CB)} B$.

Proposition:

(1) TFAE

(i) $tp(\bar{a}/B)$ is weakly one-based,

(ii) $\exists \bar{a}' \equiv_B \bar{a}$ such that $\bar{a} \downarrow_B \bar{a}'$, $\bar{a} \downarrow_{\bar{a}'} B$ and $\bar{a}' \downarrow_{\bar{a}} B$,

(iii) $tp(\bar{a}/B)$ is weakly locally modular.

(2) TFAE

(i) every independent extension of $tp(\bar{a}/B)$ is weakly one-based,

(ii) every independent extension of $tp(\bar{a}/B)$ is very weakly one-based,

(iii) every independent extension of $tp(\bar{a}/B)$ is weakly locally modular.

Definition:

An almost canonical base for $tp(\bar{a}/B)$ is a subset $E \subseteq acl(B)$ such that, for all $F \subseteq acl(B)$,

$$\bar{a} \perp_F B \quad \text{iff} \quad E \subseteq acl(F).$$

(If such E exists then it is unigne up to interalgebraicity.)

Proposition: Suppose $B = \mathrm{acl}(B)$.

If every independent extension of $tp(\bar{a}/B)$ has an almost canonical base E then TFAE

(i) $\bar{a} \downarrow_{\mathrm{acl}(\bar{a}) \cap \mathrm{acl}(B)} B$,

(ii) $tp(\bar{a}/B)$ is weakly one-based,

(iii) $tp(\bar{a}/B)$ is one-based,

(iv) $tp(\bar{a}/B)$ is very weakly one-based,

(v) $tp(\bar{a}/B)$ is weakly locally modular,

(vi) $E \subseteq \mathrm{acl}(\bar{a})$.

Suppose M is geometric.

(So (1) $\forall a, b \in M$ and $C \subseteq M$, if $a \in acl(Cb) \setminus acl(C)$ then $b \in acl(Ca)$

and (2) $\forall \varphi(x, \bar{y})$, $\{\bar{b} \in M^n : \varphi(M, \bar{b})$ is finite$\}$ is definable.)

Suppose $\downarrow$ is the independence relation obtained from acl in M.

<u>Theorem (B, V)</u>: TFAE

(i) $(M, \downarrow)$ is weakly one-based,

(ii) every minimal type in M is weakly one-based,

(iii) every reduct of $(M, \downarrow)$ is weakly one-based,

(iv) M is generically linear.

Definition:

M is generically linear if,

for every parameter set $C \subseteq M$, definable set $Y \subseteq M^n$ defined over C and formula $\varphi(xz, \bar{y})$ such that

(i) $\dim(\varphi(M^2, \bar{b})) = 1 \quad \forall \bar{b} \in Y$ and

(ii) $\dim(\varphi(M^2, \bar{b}) \wedge \varphi(M^2, \bar{b}')) = 0$

$\forall\ \bar{b}, \bar{b}' \in Y$ such that $\bar{b}' \notin acl(C\bar{b})$,

$\dim(Y) \not\geq 2$.

$(\dim(Y) < 2)$

Further suppose that M is superrosy with þ-rank $= 1$ and $\downarrow$ is þ-independence in M^{eq}.

Proposition (B, V):

If $(M, \downarrow)$ is weakly one-based then M is linear.

Proposition (B, B, K, O, P):

If M is linear then M is generically linear.

Corollary: TFAE

(i) $(M, \downarrow)$ is weakly one-based,

(ii) M is generically linear,

(iii) M is linear.

Definition:

A lovely pair of models of T is an expansion $(N, P(N))$ of a model $N \models T$ such that

(i) $P(N) \prec N$,

(ii) for every infinite $X \subseteq N$ definable in N, $X \cap P(N) \neq \emptyset$,

(iii) for every infinite $X \subseteq N$ definable in N and parameter set $B \subseteq N$, $X \nsubseteq acl_N(BP(N))$.

Let $(N, P(N))$ be a sufficiently saturated lovely pair of models of T.

Theorem (B): $(N, P(N))$ is superrosy with þ-rank $\leq \omega$.

Theorem (B, V): If $(M, \downarrow)$ is weakly one-based then $p\text{-rank}((N, P(N))) \leq 2$.

$$\left(\downarrow^{scl} \wedge \downarrow^{acl} \Rightarrow \downarrow^{p} \right)$$

Proposition (B, B, K, O, P):

If M is weakly one-based then $(N, P(N))$ is weakly one-based.

Drop the superrosiness and p-rank assumptions.

<u>Theorem (B,V):</u> TFAE

(i) $(M, \downarrow)$ is weakly one-based,

(ii) acl in $(N, P(N))$ agrees with acl in N,

(iii) $\downarrow^{scl}$ in $(N, P(N))$ is modular:

$$\bar{a} \downarrow^{scl}_{scl(\bar{a}) \cap scl(B)} B.$$