Model theory of the adeles

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Adeles

K a global field of characteristic 0 (so $[K : \mathbb{Q}] < \omega$)
To K we attach $\mathbb{A}_K$, the **adeles of** K, a locally compact commutative ring with 1.

$\mathbb{A}_K$ is a *restricted product* (in a sense to be described below) of the family of all completions $\{K_p\}$ of K at prime divisors p

[see Cassels-Frohlich (Tate thesis) for this convenient notation]

- K_p may be $\mathbb{R}$: $|\cdot|_p$ usual absolute value
- K_p may be $\mathbb{C}$: $|\cdot|_p$ square of usual absolute value
- K_p may be p-adic: $|x| = (Np)^{-\nu_p(x)}$ where Np=cardinal of residue field of ν_p

Unit ball

$$\mathcal{O}_p = \{x \in K_p : |x|_p \leq 1\}, \quad \text{compact}$$

Write $\mathcal{P}$ for maximal ideal in the p-adic (nonarchimedean case)

Restricted product

$\mathbb{A}_K$ is a subring of $\prod_p K_p$, consisting of the f such that $\{p : f(p) \notin \mathcal{O}_p\}$ is finite.

$K \to \mathbb{A}_K$ *via*
$\alpha \to$ constant function α

Topology
The K_p have the standard locally compact metric topologies.
$\mathbb{A}_K$ has as a basis of open sets the products $\prod_p U_p$, where U_p is open and equal to $\mathcal{O}_p$ for all but finitely many p

Measure
K_p has Haar measure μ_p normalised so $\mu_p(\mathcal{O}_p) = 1$.
μ_K (or μ if K understood) is Haar measure with $\mu_K(\prod \mathcal{O}_p) = 1$.

The definable sets

We consider sets definable in the ring language, either in $\mathbb{A}_K$ for fixed K, or in $\mathbb{A}_K$ for varying K.
For such sets we consider

- their topological structure
- measurability
- measure

via **quantifier elimination**.

Method: lift from the K_p by method of Feferman-Vaught internalised using Boolean algebra of idempotents of $\mathbb{A}_K$.
Boolean algebra $\mathbb{B}_K$

$$\begin{aligned} \mathbb{B}_K &= \{e \in \mathbb{A}_K : \ e^2 = e\} \\ e \wedge f &= ef \\ \neg e &= 1 - e \\ e \vee f &= e + f - ef \end{aligned}$$

definable in $\mathbb{A}_K$

Cont

Let P be the set of all $\mathcal{P}$.
Then the set of idempotents of $\mathbb{A}_K$ corresponds to *powerset*(P), even as boolean algebras, via

$$e \rightarrow \{pe(p) = 1\}$$

In Feferman-Vaught theory one considers, for ring formulas $\Phi(\nu_1, ..., \nu_n)$ and $f_1, ..., f_n \in \mathbb{A}_K$

$$(\#) \qquad [[\Phi(f_1, ..., f_n)]] = \{p : K_p \models \Phi(f_1(p), .., f_n(p)\} \in powerset(P)$$

and this naturally corresponds to an idempotent

Essential point 1

For fixed Φ, the map

$$\mathbb{A}_K \rightarrow \textit{idempotents}$$

given by (#) is definable in the ring language (even uniformly in K)

A basic ingredient is the correspondence

$$p \rightarrow e_p, \quad e_p(p) = 1, \quad e_p(q) = 0 \textit{ for } q \neq p$$

from P to **minimal idempotents**

Essential point 2

The map $\mathbb{A}_K \to \mathbb{A}_K$ given by

$$x \to e_p \cdot x$$

has kernel $(1 - e_p)\mathbb{A}_K$, and image

$$\begin{array}{ccc} e_p \cdot \mathbb{A}_K & \cong & K_p \\ e_p \cdot x & \leftarrow & x \end{array}$$

Both points are not specific to the use of the K_p, but the next is.

Fact: Uniformly in K and for all p which are not complex, there is a ring-theoretic definition of $\mathcal{O}_p$ (*topology uniformly definable*)

Consequence: uniformly in K one can first-order define the finite idempotents, i.e. those e which are the union of finitely many minimal idempotents (call this set **FIN**)

Feferman-Vaught, first version

Boolean formalism on $\mathbb{B}_K$
Usual $\wedge$, $\vee$, $\neg$, 0, 1 predicates

- $card(e) \leq n$, meaning e has $\leq n$ atoms below it
- $FIN(e)$, meaning e is a finite idempotent

Fact (1950's - Tarski or Vaught)
$\mathbb{B}_K$ has Q.E. in above formalism

Recall, for $\Phi(\nu_1, \ldots, \nu_n)$ a ring formula, the map $[[\Phi]] : \mathbb{A}_K^n \to \mathbb{B}_K$

Theorem
For every ring formula $\Phi(\nu_1, \ldots, \nu_n)$ there are (effectively) ring formulas $\Phi_1(\bar{\nu}), \ldots, \Phi_r(\bar{\nu})$ and a $\Psi(w_1, \ldots, w_r)$ from Boolean formalism so that for all K

$$\mathbb{A}_K \models \Phi(\bar{\nu}) \iff \Psi([[\Phi_1]](\bar{\nu}), \ldots, [[\Phi_r]](\bar{\nu}))$$

Feferman-Vaught, cont

To be useful in applications we need to get $\Phi_1, ..., \Phi_r$ of a simple form, and this requires **quantifier elimination** for the K_p.

This we have for fixed K using work of various authors. An essential role is played by solvability predicates $SOL_n(x_1, ..., x_n)$ expressing (in the K_p) that $x_1, ..., x_n \in \mathcal{O}_p$ and

$$y^n + x_1 y^{n-1} + .. + x_n \quad \textit{is solvable in } \mathcal{O}_p/p.$$

[This in turn relates to Riemann hypothesis for curves and ultimately to motivic issues]

Consequences

-Every definable set is Borel (but need to be locally closed)

-Each $\mathbb{A}_K$ is decidable (Weisspfenning, 1970's)

Example of a definable set not a finite union of locally closed sets

$$X = \{f : FIN([x^2 \neq x](f))\}$$

Let $\qquad X^{(1)} = fr(fr(X)) \qquad$ (fr=frontier)

Then $X^{(1)} = X$, and result follows from work of Miller and Dougherty.

In fact, X is not in $F_\sigma \cap G_\delta$, by work of Hausdorff.

X is actually F_σ and not G_δ. The following *locates* definable sets in the bottom reaches of the Borel hierarchy

Basic definable sets and their places in Borel hierarchy

1. $\{\bar{f} : [[\Phi(\bar{\nu})]](\bar{f}) = 0\}$, Φ a ring formula, is a finite union of locally closed sets
2. Same with $[[\Phi(\bar{\nu})]](\bar{f}) = 1$.
3. $\{\bar{f} : FIN[[\Phi(\bar{\nu})]](\bar{f}) = 0\}$ is a countable union of locally closed sets
4. $\{\bar{f} : \neg FIN[[\Phi(\bar{\nu})]](\bar{f}) = 0\}$ is a countable intersection of locally closed sets

Uniformity in K

Basic limitation on our knowledge:

- we do not have a uniform Q.E. for all K_p
- we do not know decidability of the class of all K_p
- for fixed p, we do not know decidability of the class of all finite extensions of $\mathbb{Q}_p$

The problem is unbounded ramification

Theorem

If the third problem is decidable, so is the second

This follows from the preceding, and the following results, due to Raf Cluckers and separately to Jamshid Derakhshan and me

Restricted effect of ramification

Theorem

There is an effective procedure which to any ring sentence Φ attaches

1. *A prime p_0*
2. *a ring sentence Φ^**

so that for any K, p such that the residue field has characteristic $p \geq p_0$

$$K_p \models \Phi \iff \textit{residue field} \models \Phi^*$$

Computing measures - Case $K = \mathbb{Q}$

Fix $n > 0$.
Let X consist of the adeles f such that

$$|f(\mathbb{R})|_{\mathbb{R}} \leq 1 \quad \text{and} \quad 0 \leq \nu_p(f(p)) \leq n$$

at the primes.
Then the measure of X is $\frac{1}{\zeta(n+1)}$

For general **rectangles** as above, one must use the Denef-Loeser work on motivic integration (work in *slow* progress)

Remarks on "stable embedding"

The individual $\nu_p : K_p \to \mathbb{Z} \cup \{\infty\}$ induce a product

$$\nu : \mathbb{A}(K) \to \prod_p (\mathbb{Z} \cup \{\infty\}).$$

It is more natural to consider ν restricted to $\{f : [[f = 0]] = 0\}$ and taking values in the lattice ordered group Γ, where $\Gamma =$ *the subgroup of* $\prod_p \mathbb{Z}$ *consisting of the* g *with* $g(p) \geq 0$ *for almost all* p

Theorem

(i) Γ *satisfies a Peano Axiom saying that each* $\{\gamma : \gamma \geq a\}$ *is well-founded for definable sets.*

(ii) Γ *is interpretable in* $\mathbb{A}(K)$

(iii) Γ *gets only its pure lattice-ordered abelian structure inside* $\mathbb{A}(K)$

Second remark

Theorem

Each K_p *is stably embedded in* $\mathbb{A}_K$.

[*Recall*: $K_p = e_p \cdot \mathbb{A}_K$]

Enriched language

We noted (as others surely have over the last 60 years) that if we enrich the Boolean structure further by adding for $n \geq 2$ a predicate $FIN_{n,r}$ to mean *has cardinality congruent to r mod n* we still have quantifier elimination and decidability.

This gives the obvious corresponding results in the adelic situation. Though we have not verified it in this situation, we expect that the extended formalism has more expressive power than the original ring formalism.