Around domination, orthogonality and parallelism in superstable MAECs

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Abstract

In this work we present some results on Geometric Stability Theory in superstable *Metric Abstract Elementary Classes.*

Metric AECs

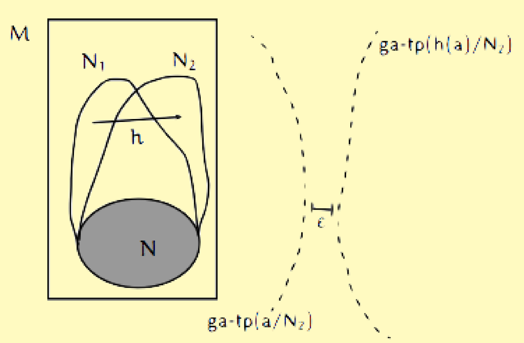
Metric Abstract Elementary Classes (shortly MAEC) is the metric version of the notion of *Abstract Elementary Classes.*

Definition 0.1. Let $\mathcal{K}$ be a class of L-structures (in the context of Continuous Logic) and $\prec_{\mathcal{K}}$ be a binary relation defined in $\mathcal{K}$. We say that $(\mathcal{K}, \prec_{\mathcal{K}})$ is a *Metric Abstract Elementary Class* (shortly *MAEC*) if:

1. $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under isomorphism.
2. $\prec_{\mathcal{K}}$ is a partial order in $\mathcal{K}$.
3. If $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
4. (*Completion of Union of Chains*) If $(M_i : i < \lambda)$ is a $\prec_{\mathcal{K}}$-increasing chain then
   (a) the function symbols in L can be uniquely interpreted on the completion of $\bigcup_{i<\lambda} M_i$ in such a way that $\overline{\bigcup_{i<\lambda} M_i} \in \mathcal{K}$
   (b) for each $j < \lambda$, $M_j \prec_{\mathcal{K}} \overline{\bigcup_{i<\lambda} M_i}$
   (c) if each $M_i \prec_{\mathcal{K}} N$, then $\overline{\bigcup_{i<\lambda} M_i} \prec_{\mathcal{K}} N$.
5. (*Coherence*) if $M_1 \subseteq M_2 \prec_{\mathcal{K}} M_3$ and $M_1 \prec_{\mathcal{K}} M_3$, then $M_1 \prec_{\mathcal{K}} M_2$.
6. (DLS) There exists a cardinality $LS^d(K)$ (which is called the *metric Löwenheim-Skolem number*) such that if $M \in \mathcal{K}$ and $A \subseteq M$, then there exists $N \in \mathcal{K}$ such that $dc(N) \leq dc(A) + LS^d(K)$ and $A \subseteq N \prec_{\mathcal{K}} M$.

ε-splitting and s-independence

Definition 0.2 (ε-splitting). Let $N \prec_{\mathcal{K}} M$ and $\varepsilon > 0$. We say that ga-tp(a/M) *ε-splits* over N iff there exist N_1, N_2 with $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$ and $h : N_1 \cong_N N_2$ such that $\mathbf{d}(\text{ga-tp}(a/N_2), h(\text{ga-tp}(a/N_1))) \geq \varepsilon$. We use $a \perp^{\varepsilon}_{N} M$ to denote the fact that ga-tp(a/M) does not ε-split over N.



Definition 0.3. Let $N \prec_{\mathcal{K}} M$. Fix $\mathcal{N} := \langle N_i : i < \sigma \rangle$ a resolution of N. We say that a is *s-independent* from M over N relative to $\mathcal{N}$ (denoted by $a \perp^{\mathcal{N}}_{N} M$) iff for every $\varepsilon > 0$ there exists $i_\varepsilon < \sigma$ such that $a \perp^{\varepsilon}_{N_{i_\varepsilon}} M$.

Properties s-Independence

Under stability, s-independence satisfies nice properties (see [ViZa1])

1. Invariance: If $f \in Aut(\mathbb{M})$ and $a \perp^{\mathcal{N}}_{N} M$, then $f(a) \perp^{f[\mathcal{N}]}_{f[N]} f[M]$
2. Motonicity: If $a \perp^{\mathcal{M}_0}_{M_0} M_3$, $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2 \prec_{\mathcal{K}} M_3$ and $\mathcal{M}_0 \prec_{\mathcal{K}} \mathcal{M}_1$, then $a \perp^{\mathcal{M}_1}_{M_1} M_2$.
3. Stationarity: If $a \perp^{\mathcal{N}}_{N} M$, M is universal over N and $M \prec_{\mathcal{K}} M'$, there exists $b \models$ ga-tp(a/M) such that $b \perp^{\mathcal{N}}_{N} M'$, and this extension is unique.
4. Transitivity: Let $M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} M_2$ be such that M_1 and M_0 are limit over some $M' \prec_{\mathcal{K}} M_0$ (are witnessed by $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively, where $\mathcal{M}_0 \subset \mathcal{M}_1$). Then $a \perp^{\mathcal{M}_0}_{M_0} M_2$ iff $a \perp^{\mathcal{M}_0}_{M_0} M_1$ and $a \perp^{\mathcal{M}_1}_{M_1} M_2$.
5. Continuity: If $(b_n) \to b$ in $\mathbb{M}$ and $b_n \perp^{\mathcal{N}}_{N} M$ for all $n < \omega$, then $b \perp^{\mathcal{N}}_{N} M$.

Superstability in first order theories

Fact 0.4. *Given T a first order theory, the following are equivalent:*

1. *T is superstable*
2. $\kappa(T) = \aleph_0$
3. *Union of an ≺-increasing chain of saturated models is saturated.*
4. *Uniqueness (up to isomorphism) of limit models.*

Superstability-like assumptions

Assumption 0.5 (superstability). *For every a and every increasing and continuous $\prec_{\mathcal{K}}$-chain of models $\langle M_i : i < \sigma \rangle$ and $\mathcal{M}_j$ a resolution of M_j $(j < \sigma)$:*

1. *(continuity) If $p \restriction M_i \perp^{\mathcal{M}_0}_{M_0} M_i$ for all $i < \sigma$, then $p \perp^{\mathcal{M}_0}_{M_0} \overline{\bigcup_{i<\sigma} M_i}$.*
2. *(locality) If $cf(\sigma) > \omega$, there exists $j < \sigma$ such that $a \perp^{\mathcal{M}_j}_{M_j} \bigcup_{i<\sigma} M_i$.*
3. *(ε-simplicity) If $cf(\sigma) = \omega$, there exists $j < \sigma$ such that $a \perp^{\varepsilon}_{M_j} \overline{\bigcup_{i<\sigma} M_i}$.*

Fact 0.6 (Uniqueness of Limit Models). *If M_i is a (μ, θ_i)-d-limit $(i \in \{1, 2\})$ over M where $dc(M_1) = dc(M_2)$, then $M_1 \cong_M M_2$.*

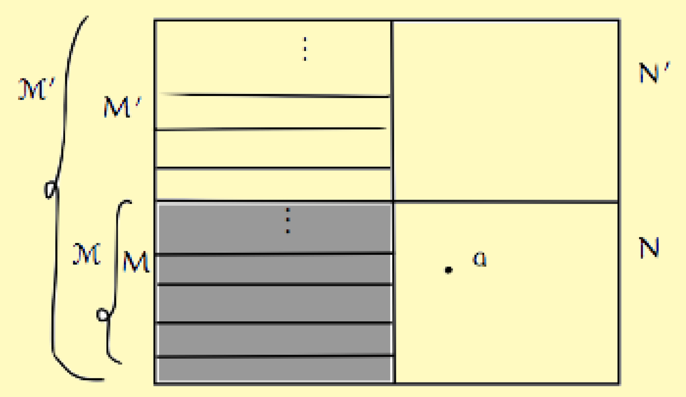
Proof. This is a consequence of assumption 0.5, following the same sketch of the proof given in [GrVaVi], but we have to point out that the details of the proof in this setting are quite different (see [ViZa2]). The key idea is to prove that given any $\theta < \mu^+$, there exists a (μ, θ)-limit model which is also a (μ, ω)-limit model. □

Domination (1)

Throughout the rest of this poster, we suppose assumption 0.5, so uniqueness of limit models holds (fact 0.6).

Notation 0.7. $(M, \mathcal{M}, N, a)$ means that $M \prec_{\mathcal{K}} N$, M is a limit model witnessed by $\mathcal{M}$ and $a \in N \setminus M$.

Definition 0.8. We say that $(M, \mathcal{M}, N, a) \prec_{nf} (M', \mathcal{M}', N', a)$ iff M' is a limit model over M, $\mathcal{M} \subset \mathcal{M}'$ and $\mathcal{M}$ corresponds to an initial segment of $\mathcal{M}'$), $N \prec_{\mathcal{K}} N'$ and $a \perp^{\mathcal{M}}_{M} M'$.



Definition 0.9. Given $(M, \mathcal{M}, N, a)$, we say that a *dominates* N over M relative to $\mathcal{M}$ (denoted by $a \rhd^{\mathcal{M}}_{M} N$) iff for every $(M', \mathcal{M}', N', a) \succ_{nf} (M, \mathcal{M}, N, a)$ we have that $N \perp^{\mathcal{M}}_{M} M'$ (i.e., for every $b \in N$ $b \perp^{\mathcal{M}}_{M} M'$).

Domination (2)

Proposition 0.10. *Given $(M, \mathcal{M}, N, a)$ there exists $(M' \mathcal{M}', N', a) \succ_{nf} (M, \mathcal{M}, N, a)$ such that $a \rhd^{\mathcal{M}'}_{M'} N'$.*

Proof. Suppose not. This allows us to construct an $\prec_{nf}$-increasing and continuous sequence of tuples $\langle (M^\alpha, \mathcal{M}^\alpha, N^\alpha, a) : \alpha < \mu^+ \rangle$ such that $(M^0, \mathcal{M}^0, N^0, a) := (M, \mathcal{M}, N, a)$ and $(M^{\alpha+1}, \mathcal{M}^{\alpha+1}, N^\alpha, a)$ witnesses that $(M^\alpha, \mathcal{M}^\alpha, N^\alpha, a)$ does not satisfy that $a \rhd^{\mathcal{M}^\alpha}_{M^\alpha} N^\alpha$. Using locality (assumption 0.5 2.), continuity and monotonicity of s-independence, we get a contradiction. □

Proposition 0.11. *Suppose $(M, \mathcal{M}, N, a) \prec_{nf} (M', \mathcal{M}', N', a)$, where M is a limit model (witnessed by $\mathcal{M} := \{M_i : i < \sigma\}$) and M' is a limit model over M (witnessed by $\mathcal{M}''$), $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$ for some limit $\alpha < \sigma$ and $a \rhd^{\mathcal{M}'}_{M'} N'$, where $\mathcal{M}' := \mathcal{M} \cup \mathcal{M}''$. Then, there exist N^* and a resolution $\mathcal{M}^*$ which witnesses that M is a limit model over M_0 such that $a \rhd^{\mathcal{M}^*}_{M} N^*$.*

Proof. Let $p :=$ ga-tp(a/M) and $p' :=$ ga-tp(a/M'). It is straightforward to see that $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M'$. Notice that M and M' are limit over $M_{\alpha+1} \succ_{\mathcal{K}} M_\alpha$. By fact 0.6, there exists $f : M' \cong_{M_{\alpha+1}} M$. By invariance, $f(a) \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$. Notice that $M_{\alpha+1}$ is universal over M_α. Therefore, since ga-tp$(a/M_{\alpha+1})$ = ga-tp$(f(a)/M_{\alpha+1})$ and $a, f(a) \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$, by stationarity we may say ga-tp(a/M) = ga-tp$(f(a)/M)$. Let $g \in Aut(\mathbb{M}/M)$ be such that $(g \circ f)(a) = a$. Notice that

$$(g \circ f)(M', \mathcal{M}', N', a) = (M, (g \circ f)[\mathcal{M}'], (g \circ f)[N'], a)$$

satisfies $a \rhd^{\mathcal{M}^*}_{M} N^*$, where $N^* := (g \circ f)[N']$ and $\mathcal{M}^* := (g \circ f)[\mathcal{M}'] = f[\mathcal{M}']$. □

Domination (3)

Corollary 0.12. *Given $(M, \mathcal{M}, a, N)$ such that $a \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$ for some $M_\alpha \in \mathcal{M}$, there exist N^* and a resolution $\mathcal{M}^*$ which witnesses that M is a limit model over M_0 such that $a \bowtie^{\mathcal{M}^*}_{M} N^*$.*

Weak Orthogonality

Definition 0.13. Let M be a limit model witnessed by $\mathcal{M} := \{M_i : i < \theta\}$, $p, q \in$ ga-S(M) be non-algebraic types such that $p, q \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$ for some limit $\alpha < \theta$ where $\mathcal{M}_\alpha \subset \mathcal{M}$. We say that p is *weak orthogonal to* q (denoted by $p \perp^{wk} q$) iff given $(M, \mathcal{M}, N, b)$ where $b \models q$ and $p' \in$ ga-S(N) any extension of p, then $p' \perp^{\mathcal{M}}_{M} N$

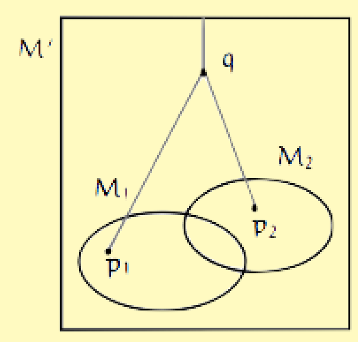
Strong Limit types

Definition 0.14 (strong limit type). Let M be a σ-limit model

$$\mathfrak{SL}(M) := \left\{ (p, N) : \begin{array}{l} N \prec_{\mathcal{K}} M \\ N \text{ is a } \theta\text{-limit model} \\ M \text{ is a limit model over } N \\ p \in \text{ga-S}(M) \text{ is non-algebraic} \\ \text{and } p \perp^{\mathcal{N}}_{N} M \\ \text{for some resolution } \mathcal{N} \text{ of N.} \end{array} \right\}$$

Parallelism of Strong Limit types (1)

Definition 0.15 (Parallelism). Two strong types $(p_1, N_1) \in \mathfrak{SL}(M_l)$ $(l \in \{1, 2\})$ are said to be *parallel* (which we denote by $(p_1, N_1) \parallel (p_2, N_2)$) iff for every $M' \succ_{\mathcal{K}} M_1, M_2$ with density character μ, there exists $q \in$ ga-S(M') which extends both p_1 and p_2 and $q \perp^{\mathcal{N}_l}_{N_l} M'$ $(l \in \{1, 2\})$ (where $\mathcal{N}_l$ is the resolution of N_l which satisfies $p_l \perp^{\mathcal{N}_l}_{N_l} M_l$). If there is no any confusion, we denote it by $p_1 \parallel p_2$.



Parallelism of Strong Limit types (2)

Fact 0.16. *∥ is an equivalence relation.*

Fact 0.17.
1. *Given $p, q \in$ ga-S(M), $\mathcal{M}$ a resolution of M which witnesses that M is a limit model such that $p, q \perp^{\mathcal{M}_\alpha}_{M_\alpha} M$ and $f : M \cong N$ is an isomorphism, then $p \perp^{wk} q \Leftrightarrow f(p) \perp^{wk} f(q)$.*
2. *If $N \succ_{\mathcal{K}} M$ is limit over M (and in particular over $M_{\alpha+1}$), given $p, q \in$ ga-S(N) such that $p, q \perp^{\mathcal{M}_\alpha}_{M_\alpha} N$, $p \perp^{wk} q$ iff $p \restriction M \perp^{wk} q \restriction M$.*
3. *If $N \succ_{\mathcal{K}} M$ is limit over M (and in particular over M_1) and $p_1, p_2 \in$ ga-S(M) and $q_1, q_2 \in$ ga-S(N) satisfy $p_i \parallel q_i$ $(i \in \{1, 2\})$, then $p_1 \perp^{wk} p_2$ iff $q_1 \perp^{wk} q_2$.*

Proof of 2. Since M and N are limit models over $M_{\alpha+1}$, by corollary 0.6 there exists $f : M \cong_{M_{\alpha+1}} N$. Notice that $p \restriction M_{\alpha+1} = f(p \restriction M_{\alpha+1}) \subset f(p \restriction M)$ and $q \restriction M_{\alpha+1} = f(q \restriction M_{\alpha+1}) \subset f(q \restriction M)$. Since $p \restriction M_{\alpha+1} = f(p \restriction M_{\alpha+1}) \perp^{\mathcal{M}_\alpha}_{M_\alpha} M_{\alpha+1}$ and $f(p \restriction M) \supset p \restriction M_{\alpha+1}$ satisfies $f(p \restriction M) \perp^{\mathcal{M}_\alpha}_{M_\alpha} N$ (by monotonicity and invariance) and $p \perp^{\mathcal{M}_\alpha}_{M_\alpha} N$, then by stationarity (notice that $M_{\alpha+1}$ is universal over M_α) we have that $f(p \restriction M) = p$. In a similar way we can prove $f(q \restriction M) = q$. By fact 0.17 (1) we have $p \restriction M \perp^{wk} q \restriction M$ iff $p \perp^{wk} q$. □

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