Preservation of Integrality Conditions on Domains under Derivations

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Introduction

This poster asks if given a derivation of a fraction field of a domain that sends the domain to itself, when and what integrality conditions over the domain does the derivation preserve. We show that this question is linked to the power series ring over the domain and in fact, give a criteria for preservation based on this power series ring.

Notation and Conventions

Let R be an integral domain containing the rationals, K its field of quotients and $\overline{R}$ its integral closure in K. Let δ be a derivation on K such that $\delta(R) \subseteq R$. Also, t shall always be an indeterminate over the ring or field in question.

Classical Integrality

In general δ need not preserve integrality as the following counterexample, due to Seidenberg, shows

Counterexample of $\alpha \in \overline{R} \Rightarrow \delta(\alpha) \in \overline{R}$ (cf. [3])

Let V be a valuation ring of rank 2 containing the rationals and let $R := V[[t]]$. Then there exists some nonunit $b \in V$ such that $\bigcap_{n\geq 1}(b^n) \neq 0$. Now let

$$\alpha = (b^2+t)^{1/2} = b[1 + c_1\frac{t}{b^2} + c_2\frac{t^2}{b^4} + \ldots]$$

where $c_1, c_2, \ldots$ are rationals. It is clear that α is integral over R. If we take $\delta = \partial/\partial t$ then $\delta(R) \subseteq R$. However, $\delta(\alpha) = 1/2(b^2+t)^{-1/2}$ is not integral over R.

However, the derivation δ does preserve integrality if in addition we have the condition that $\overline{R}[[t]]$ is integrally closed i.e.-

Theorem A (cf. [3, pg. 170])

If $\overline{R}[[t]]$ is integrally closed, then $\delta(\overline{R}) \subseteq \overline{R}$.

Power Series

In fact, it is possible to generalize Theorem A. Consider an unary predicate, $\mathcal{P}$, defined on fields which is defined a such away that given a domain, $\mathcal{P}$ is uniquely determined on its quotient field. We denote $\mathcal{P}(K) = R^{\mathcal{P}}$ and call it the *closure w.r.t.* $\mathcal{P}$. Then we have the following generalization of Theorem A-

Theorem B

If $\mathcal{P}$ is preserved under the **Taylor morphism** (w.r.t. δ) restricted to K and $(R[[t]])^{\mathcal{P}} \subseteq R^{\mathcal{P}}[[t]]$ is $\mathcal{P}$-integrally closed then $\delta(R^{\mathcal{P}}) \subseteq R^{\mathcal{P}}$

The Taylor morphism mentioned in Theorem B above is the map $E : K \to L$, where L is the quotient field of $R[[t]]$, defined by $E(\alpha) = \alpha + t\delta(\alpha) + (\frac{t^2}{2!})\delta^2(\alpha) + \ldots$. To see that E indeed maps K to L we first notice that since $\delta(R) \subseteq R$ and R contains the rationals, $E(R) \subseteq R[[t]]$. Secondly E, when considered as a map to $K[[t]]$, is an injective homomorphism; thus if $\alpha \in K, \alpha = a/b$ where $a, b \in R$ then $E(\alpha) = E(a)/E(b) \in L$. Thus E (as a map to L) is an **injective ring homomorphism**. Note that E can actually be extended to an isomorphism of $K((t))$.

Proof of Theorem B: Let $\alpha \in K$. If $K \models \mathcal{P}(\alpha)$, then $L \models E(\alpha)$ which means that $E(\alpha) \in (R[[t]])^{\mathcal{P}}$. Thus $E(\alpha) \in R^{\mathcal{P}}[[t]]$. But the coefficient of t is $\delta(\alpha)$ and hence $\delta(\alpha) \in R^{\mathcal{P}}$.

It is clear to see that Theorem A is a corollary of Theorem B. An interesting result in relation to this is the following-

Theorem (cf. [3, pg. 170])

Let R be an integral domain whose integral closure is Noetherian and is a finitely generated R-module. Then the integral closure of $R[[t]]$ is $\overline{R}[[t]]$.

Generalized Notions of Integrality

Theorem B gives us a useful tool to study other notions of integrality. We look at the following examples.

Other Notions of Integrality

An element $\alpha \in K$ is

1. **Almost Integral** over R if there exists some $b \in R$ such that $b\alpha^n \in R$ for all natural numbers n (cf. [2]).
2. **Ω-almost integral** over R if for all $b \in R$ such that $b\alpha \in R$ there exists some $m_b \in \mathbb{N}$ such that $b^{m_b}\alpha^n \in R$ for all $n \in \mathbb{N}$ (cf. [1]).
3. **m-almost integral** over R, for some $m \in \mathbb{N}$, if the $m_b = m$ for all such b in 2 (cf. [1]).

Properties of Almost Integrality Conditions

1. Integral$\Rightarrow$m-Almost Integral$\Rightarrow$ Ω-Almost Integral$\Rightarrow$Almost Integral
2. All of these conditions are preserved under the Taylor morphism.
3. $(R[[t]])^{al} \subseteq R^{al}[[t]]$, where R^{al} denotes the closure w.r.t almost integrality (cf. [3, pg. 170]).
4. When $R^{\Omega-al}$ (resp. R^{m-al}) is a ring then $(R[[t]])^{\Omega-al} \subseteq R^{\Omega-al}[[t]]$ (resp. $(R[[t]])^{\Omega-al} \subseteq R^{\Omega-al}[[t]]$).

Corollary

1. $\delta(R^{al}) \subseteq R^{al}$
2. When $R^{\Omega-al}$ (resp. R^{m-al}) is a ring, then $\delta(R^{\Omega-al}) \subseteq R^{\Omega-al}$ (resp. $\delta(R^{m-al}) \subseteq R^{m-al}$)

Other Properties Integrality Conditions (cf. [1] and [2])

1. $\alpha \in K$ is almost integral over R iff there exists some finitely generated R-submodule of K that contains all powers of α.
2. R^{al} is a ring and is often called the *complete* integral closure.
3. If R is Noetherian then almost integrality is equivalent to integrality.
4. If $R \subseteq T$ is a ring extension where every $t \in T$ is Ω-integral over every intermediate extension A that is integrally closed in T then the extension $R \subseteq T$ has the "going up" property (cf. [1, Theorem 2.13]).

However there are many failings of these integrality conditions-

Difficulties in Almost Integrality (cf. [1])

1. $R^{al}, R^{\Omega-al}$ and R^{m-al} are not closed under almost, Ω-almost and m-almost integrality respectively.
2. $R^{\Omega-al}$ and R^{m-al} need not be rings!
3. None the inclusions from R into $R^{al}, R^{\Omega-al}$ or R^{m-al} necessarily have going up or going down (thus does not preserve Krull dimension)

Open Problems and Questions

- Is there a notion of integrality which δ preserves and preserves Krull Dimension?
- If $\overline{R}$ is closed under all such derivations δ of K, is $\overline{R}[[t]]$ integrally closed?
- Under what other conditions is the ring $\overline{R}[[t]]$ integrally closed?
- Does δ preserve Ω-integrality or m-integrality when their closures are not a ring?
- What more can be understood about the connection between derivatives and integrality conditions by the studying the Taylor morphism?

References

[1] Coykendall, J., Dutta, T.: *A generalization of integrality*, submitted,

[2] Krull, W.: *Allgemeine Bewertungstheorie*, Reine Angew. Math. 167 (1932), 160-196.

[3] Seidenberg, A.: *Derivations and Integral Closure*, Pacific Journal of Mathematics, 16, No. 1, 167-173,(1966)