NOTES ON MODEL THEORY OF COMPACT LIE GROUPS

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MOTIVATIONS and BACKGROUND

Our setting

One of the most fascinating topics in Model Theory is exponentiation.
In the last years, model theorists have reached a very satisfactory understanding of the exponential function in the ordered field $\mathbb{R}$ of real numbers and approached the same question over the field $\mathbb{C}$ of complex numbers, in terms of definability, decidability and model completeness.

Motivations

Actually, exponentiation can be extended to larger settings, such as Lie algebras. Macintyre examines in [M] how to introduce exponentiations on finite-dimensional Lie algebras both over the real and complex field. This suggests to explore exponentiation also over some infinite dimensional Lie algebra [LMP]. Another matter related to the exponentiation is the study of Lie groups with the logarithmic function introduced as the inverse function of the exponential. A major nuisance for a systematic first-order work on the groups is the nonexplicit nature of the logarithmic regions, that is, natural domains for the logarithm. Macintyre also shows the logarithm suitably restricted on the real Lie group of all orthogonal matrices with determinant 1 is Pfaffian and gives a decidability result for the structure expanding by the logarithm.

Aims

- Our present aim in [LPT] is to generalize [M] to every compact Lie group, by introducing new logarithmic maps on some compact Lie groups and by investigating the issue of decidability on the related structures.The classical compact (real) Lie groups we are interested in are the following (where $n \in \mathbb{N} \geq 2$ where $\mathbb{N}$ denotes the set of natural numbers):

1. the special unitary group $SU(n)$ of all $n \times n$ unitary complex matrices with determinant 1;
2. the special orthogonal group $SO(n)$ of all $n \times n$ orthogonal real matrices with determinant 1;
3. the symplectic group $Sp(n)$ defined as the quaternionic unitary group.

Preliminaries

Let G denote one of the Lie groups $SU(n),\ SO(n),\ Sp(n)$. G is the image by the matrix exponential map $exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ of its related Lie algebra. The following result will be useful for our purposes.

FACT 1. Let n be an integer ≥ 2. Then the following holds:

(1) Every matrix $a \in SU(n)$ is conjugated to a diagonal matrix

$$a' = \text{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$$

with the θ_k $(1 \leq k \leq n)$ real number and $\prod_{1 \leq k \leq n} e^{i\theta_k} = 1$.

(2) Every matrix $a \in SO(n)$ is conjugated to a block-diagonal matrix of the form $\text{diag}(t_1, \ldots, t_m)$ if $n = 2m$ is even, and $\text{diag}(t_1, \ldots, t_m, 1)$ if $n = 2m+1$ is odd, where each t_j $(j = 1, \ldots m)$ is a 2×2 block

$$\begin{pmatrix} cos(\theta_j) & -sin(\theta_j) \\ sin(\theta_j) & cos(\theta_j) \end{pmatrix}$$

with θ_j a real. Moreover the 1 for $n = 2m+1$ is a 1×1 block.

(3) Every matrix $a \in Sp(n)$ is conjugated over the complexes to a diagonal matrix $2n \times 2n$ of the form $a' = \text{diag}(e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_n}, e^{-i\theta_n})$ with the θ_k $(1 \leq k \leq n)$ real numbers.

For instance, an element a in $SU(2)$ is conjugate to a matrix a' of the form $\text{diag}(e^{i\theta},\ e^{-i\theta})$ with $\theta \in \mathbb{R}$.

Recall that a logarithm map for a matrix $a \in M(n, \mathbb{C})$ -so an inverse function of the exponential- can be defined by the power series $log(a) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}(a - I_n)^k$.

However there are strong bounds on the domain of this map.
For example, in $SU(2)$ the logarithm of diag $\left(e^{i\theta},\ e^{-i\theta}\right)$ is defined when the series converges, so whenever $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$. We aim at extending this definition of logarithm (by using the quoted properties of matrices in our groups).

MAIN RESULTS

Defining logarithm

We define for unitary, orthogonal or symplectic matrices a function Log. Let us distinguish these three cases.

Case 1: $SU(n)$. Let $a \in SU(n)$, a' as in Proposition FACT 1, (1). Thus $a' = u^{-1}au$ for some invertible $n \times n$ matrix u and $a' = \text{diag}(e_1, \ldots, e_n)$ with $e_k = e^{i\theta_k}$ for $k = 1, \ldots, n$ and $\prod_{1 \leq k \leq n} e_k = 1$. Take $-\pi < \theta_k \leq \pi$ for every $i = 1, \ldots, n-1$ and $\sum_{1 \leq k \leq n} \theta_k = 0$. Now define $Log(a') = diag(i\theta_1, \ldots, i\theta_n)$. Then put

$$Log(a) = Log(ua'u^{-1}) := uLog(a')u^{-1}.$$

Observe that, if $b' = diag(i\theta_1, \ldots, i\theta_n)$, then $a' = exp(b')$. Then, it makes sense to put $b' = Log(a')$. Moreover
$a = u \cdot a' \cdot u^{-1} = u \cdot exp(b') \cdot u^{-1} = exp(u \cdot b' \cdot u^{-1})$. Note the u is unique up to the composition with a product of involution matrices and a is unique up to a permutation of its diagonal elements.

Case 2: $SU(n)$. Every $a \in SO(n)$ is conjugated over the reals to a matrix a' as in FACT 1, (2). Let $u \in GL(2, \mathbb{R})$ be such that $u \cdot a \cdot u^{-1} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$ and let $c \in GL(2, \mathbb{C}) = \begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}$, whence $c \cdot u \cdot a \cdot u^{-1} \cdot c^{-1} = \text{diag}(e^{i\theta},\ e^{-i\theta})$. Define $Log(a) = u^{-1} \cdot c^{-1} \cdot$ **diag**$(i\theta, -i\theta) \cdot c \cdot u$**.** a' unique up a product of involution matrices.

Case 3: $Sp(n)$. By FACT 1, (3), the same arguments as in Case 1 can be repeated.

Now consider the ordered field of reals $\overline{\mathbb{R}} = (\mathbb{R}, +, ., -, >, 0, 1)$ and expand it (to get $(\overline{\mathbb{R}},\ Log_{\upharpoonright G})$) by the logarithm map Log on a compact Lie group G. Basic trigonometry shows that this structure is strictly related from the model theoretic point of view to the expansion of $\overline{R}$ by the inverse tan^{-1} of the tangent function, or even to the expansion of $\overline{R}$ by the restriction τ of this function to the interval $[0, 1]$.
Let $\mathcal{L}_\tau$ be the language expanding that of $\overline{\mathbb{R}}$ by a 1-ary operation symbol.

Outcomes

CRUCIAL FACT. For every $G = SU(n),\ SO(n),\ Sp(n)$, the structure $(\overline{\mathbb{R}},\ Log_{\upharpoonright G})$ is (existentially) bi-interpretable with $(\overline{\mathbb{R}},\ \tau)$.

Model completeness and o-minimality

The theory T_τ of $(\overline{\mathbb{R}},\ \tau)$ is model complete and o-minimal, then by interpretability the theory T_{Log} of $(\overline{\mathbb{R}},\ Log_{\upharpoonright G})$ **is model complete and o-minimal.**

The decidability issue

We examine the decidability issue for T_{Log} by following the approach of Macintyre-Wilkie to the decision problem for the restricted exponential function in the reals [MW].

- Partial decidability results are obtained for a quantifier free fragment of $(\overline{\mathbb{R}},\ \tau)$ and $(\overline{\mathbb{R}},\ tan^{-1})$ (using an effective **Lang property** in the unrestricted case).
- Consider now the following assumption (related to the classical Schanuel Conjecture in Number Theory.

Schanuel Conjecture on the circle (SC_τ) : *Let $r_1, \ldots, r_n \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then the transcendence degree over the field of rational numbers $\mathbb{Q}$ of $\mathbb{Q}(r_1, \ldots, r_n, e^{i \cdot r_1}, \ldots, e^{i \cdot r_n}$ is $\geq n$.*

Weak Schanuel Conjecture (WSC_τ): *There exists an effective procedure which, given $r \leq n \in \mathbb{N}$ and function $\bar{f} = (f_1, \ldots, f_n)$ and g from $[0,1]^r \times \mathbb{R}^{n-r}$ into $\mathbb{R}$ which are polynomial in the unknowns $x_1, \ldots, x_n$ and $\tau(x_1), \ldots, \tau(x_r)$ in the interior of their domain, produces a non zero number $\eta := \eta(n, r, \bar{f}, g)$ such that for every non singular zero $\bar{\alpha}$ of $\bar{f}$ in $[0,1]^r \times \mathbb{R}^{n-r}$, $(\bar{f})$ satisfies either $g(\bar{\alpha}) = 0$ or $|g(\bar{\alpha})| > \eta^{-1}$.*

Theorem. WSC_τ is equivalent to the decidability of T_τ and so WSC_τ implies the decidability of the theory T_{Log}.

Theorem. Modulo SC_τ the theory T_τ is decidable, whence T_{Log} is also decidable.

Corollary. SC_τ implies WSC_τ.

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