Positive model theory in the footsteps of Ben Yaacov and Poizat

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First-order logic without negation

Restricted set of logical connectives and quantifiers :

$$\wedge \ , \ \vee \ , \ \forall$$

extended by the antilogy : $\perp$.
Positive first-order formulas in prenex normal form :

$$\exists \bar{y} \ f(\bar{x}, \bar{y})$$

with $f(\bar{x}, \bar{y})$ positive and quantifier-free.
Special set of first-order sentences : h-inductive sentences :

$$\forall \bar{x} \ \exists \bar{y} f(\bar{x}, \bar{y}) \ \rightarrow \ \exists \bar{z} g(\bar{x}, \bar{z})$$

with $f(\bar{x}, \bar{y})$, $g(\bar{x}, \bar{z})$ positive and quantifier-free. In prenex normal form,

$$\forall \bar{u} \exists \bar{v} \ \neg \phi(\bar{u}) \ \vee \ \psi(\bar{u}, \bar{v})$$

with $\phi(\bar{u})$ et $\psi(\bar{u}, \bar{v})$ quantifier-free.
Important special case, h-universal sentences : $\neg \exists \bar{u} \ \phi(\bar{u})$.

Same notions of model and consistancy, and a compactness theorem :

Theorem 1 (Ben Yaacov-Poizat (2006)) *An h-inductive theory is consistant if and only if every finite subset of it is consistant.*

A more general notion of embedding :

Definition 1 (Immersion) *An mapping h from a structure of $\mathcal{M}$ into a structure $\mathcal{N}$ is an* immersion *if for every $\bar{m}$ extracted from $\mathcal{M}$ and positive first-order formula $\phi(\bar{x})$, $\mathcal{M} \models \phi(\bar{m})$ if and only if $\mathcal{N} \models \phi(h(\bar{m}))$.*

Types

A special class of models of an h-inductive theory :

Definition 2 (PEC) *A model $\mathcal{M}$ of an h-inductive theory T is* positively existentially closed (*"pec" in short*) *if every homomorphism from $\mathcal{M}$ into another model of T is an immersion.*

Companions of an h-inductive theory T : theories having the same pec models as T.

- minimal companion : T_u, the h-universal consequences of T ;
- maximal companion : T_k, the h-inductive consequences of T, its *Kaiser envelope.*

Companions have the same pec models.
Types are realized in pec models :

Definition 3 (Type) *An n-*type *of an h-inductive theory T is the set of positive first-order formulas realized by an n-tuple $\bar{a}$ in a pec model of T. As usual, it is noted $S_n(T)$.*

Fact 1 (Ben Yaacov-Poizat (2006)) *A model $\mathcal{A}$ of an h-inductive theory T is pec if and only if for every n-tuple $\bar{a}$ from $\mathcal{A}$ the set of positive formulas satisfied by $\bar{a}$ is an n-type.*

The set $S_n(T)$ is topologized using positive formulas. It is compact but not necessarily Hausdorff.

Fact 2 (Ben Yaacov-Poizat (2006)) *Let T be an h-inductive theory. Then $S_n(T)$ is Hausdorff if and only if one can amalgamate the homomorphismes between models of the Kaiser envelope T_k : for every triple of models $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ of T_k together with homomorphisms $f : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ and $g : \mathcal{M}_1 \longrightarrow \mathcal{M}_3$, there exist $\mathcal{M}_4 \models T_k$ and homomorphisms h and k such that the following diagramme commutes :*

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{f} & \mathcal{M}_2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{M}_3 & \xrightarrow[k]{} & \mathcal{M}_4 \end{array}$$

Elementary extensions

Definition 4 (Poizat) *Let $\mathcal{M}$ be structure. A positive elementary extension $\mathcal{N}$ of M is a pec model of $T_u(M)$ (equivalently, of $T_k(M)$), denoted $\mathcal{M} \preceq_+ \mathcal{N}$.*

The notations $T_u(M)$ and $T_k(M)$ mean that we allow parameters from the underlying set of $\mathcal{M}$. Similarly for $S_n(M)$.

Fact 3 (Poizat (2008)) *If $\mathcal{M} \preceq_+ \mathcal{N}$ and $S_n(N)$ is Hausdorff then so is $S_n(M)$.*

Theorem 2 (Belkasmi (2009)) *If $\mathcal{M} \preceq_+ \mathcal{N}$ and $S_n(M)$ is Hausdorff then so is $S_n(N)$.*

Towards eliminating quantifiers : positive Robinson theories

Definition 5 (Positive Robinson theory) *An h-inductive theory T is said to be* positive Robinson *if the positive types are determined by their quantifier-free fragments.*

Such theories can be characterized using special models :

Definition 6 (h-maximal models (Kungozhin (2011)) *A model $\mathcal{M}$ of an h-inductive theory T is h-*maximal *if every homomorphisme from $\mathcal{M}$ into another model of T is an embedding.*

Theorem 3 (Belkasmi (2011)) *Let T be a positive Robinson theory. Then every model of T that embeds in a pec model of T is an h-maximal model of T. Moreover, the h-maximal models of T have the amalgamation property. If, in addition, T is h-universal, then the conditions are sufficient.*

Quantifier elimination

Definition 7 *An h-inductive theory T has the* quantifier elimination property *if, for $\mathcal{A}$ and $\mathcal{B}$ models of T, and $\bar{a}$ and $\bar{b}$ from $\mathcal{A}$ and $\mathcal{B}$ respectively, $\bar{a}$ and $\bar{b}$ satisfy the same positive formulas (in $\mathcal{A}$ and $\mathcal{B}$ respectively) if and only if they satisfy the same quantifier-free positive formulas.*

Theorem 4 (Belkasmi (2011)) *Let T be an h-universal theory. The the following are equivalent :*

1. *T has the quantifier elimination property.*
2. *Every model of T is weakly pec.*
3. *Every positive formula is equivalent modulo T to a quantifier-free positive formula.*

A model $\mathcal{M}$ of an h-inductive theory T is *weakly pec* if every embedding from $\mathcal{M}$ into another model of T is an immersion. A model of T is pec if and only if it is weakly pec and h-maximal.

What's next ?

Stability and simplicity in the context of positive logic

Références

[1] Itaï Ben Yaacov, Bruno Poizat. *Fondements de la logique positive. Journal of Symbolic Logic*, 72, 4, 1141–1162, 2007.

[2] Itaï Ben Yaacov. *Positive model theory and compact abstract theories. Journal of Mathematical Logic*, no. 1, 85–118, 2003.

[3] Bruno Poizat. *Quelques effets pervers de la positivité. Annals of Pure and Applied Logic*, no.6, 812–816,2010.

[4] Bruno Poizat. *Univers Positifs. Journal of Symbolic Logic*, 71, 3, 969–976, 2006.

[5] Anand Pillay Forking in the category of existentially closed structures *Quaderni di Matematica,6, Université de Naples.*